One-dimensional transport equation models for sound energy propagation in long spaces: Theory

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In this paper, a three-dimensional transport equation model is developed to describe the sound energy propagation in a long space. Then this model is reduced to a one-dimensional model by approximating the solution using the method of weighted residuals. The one-dimensional transport equation model directly describes the sound energy propagation in the “long” dimension and deals with the sound energy in the “short” dimensions by prescribed functions. Also, the one-dimensional model consists of a coupled set of \( N \) transport equations. Only \( N=1 \) and \( N=2 \) are discussed in this paper. For larger \( N \), although the accuracy could be improved, the calculation time is expected to significantly increase, which diminishes the advantage of the model in terms of its computational efficiency. © 2010 Acoustical Society of America. [DOI: 10.1121/1.3298936]

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I. INTRODUCTION

Long spaces, in which one dimension is much larger than the other two, are of particular interest in room acoustics, as they usually take place in tunnels, underground stations, corridors, and some factories. Speech intelligibility and noise evaluation in these public places are critically important\(^1\) and sometimes need to be predicted before construction.

The classical statistical theory, e.g., the Sabine equation, is not applicable in these situations because the sound field is highly nonuniform.\(^1,2\) Other approaches, which fall into the category of the geometrical acoustic model, have recently been studied intensively, including the image source method,\(^3–5\) the ray-tracing based method,\(^6,7\) radiosity,\(^8,9\) the diffusion equation model,\(^10–13\) and the rendering equation model.\(^14\) The image source method fails to include diffuse reflections, which have been found to be crucial when studying the steady-state and transient sound fields in long enclosures.\(^15\) In contrast, the diffusion equation and original radiosity methods only take diffuse reflection into account. Although empirical modifications\(^16\) have been applied to the diffusion equation method to model specular reflections, the model itself is inherently only suitable for low absorptive surfaces. The radiosity method can be modified to treat partially diffusely reflecting surfaces by combining the extended radiosity and mirror-image methods.\(^17\) The ray-tracing based method is able to consider specular and diffuse reflections, but it is time-consuming due to its use of the Monte Carlo method. More recently Polles \textit{et al.}\(^18,19\) touched upon the transport theory in the context of urban streets. However, they did not formulate explicit solutions of the transport equations in their modeling effort rather than resort to asymptotic solutions in the form of diffusion equations.

The purpose of this work is to introduce a family of versatile one-dimensional transport equation models, which can simulate specular and diffuse reflections, air dissipations, scattering objects inside the enclosures, and different source types, as well as source directivities. These one-dimensional transport equations are (i) derived from a fundamental three-dimensional transport equation for sound propagation in enclosures and (ii) less computationally expensive than directly solving the three-dimensional transport equation since the number of independent spatial variables is reduced from 3 to 1.

All the theoretical considerations are presented in Sec. II. Although the mathematics is based on previous work in neutral particle transport,\(^20–22\) differences are brought in by introducing several acoustic concepts. Specifically, different types of acoustic sources, boundary conditions at the two ends of a long space, and the specular reflections on side walls for a so-called two-group model are introduced for the first time in architectural acoustics.

This paper is structured as follows: first, Sec. II A presents the exact three-dimensional transport equation with corresponding boundary conditions. Then the method of weighted residuals is utilized to approximate the three-dimensional transport equation and simplify it to a coupled system of one-dimensional transport equations. Section II B discusses the details of the two simplest models, one having one one-dimensional transport equation, and the other having two coupled one-dimensional transport equations. The latter is shown through a simple test to be more accurate. Section

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II C also discusses the possibilities of modeling different sound sources and scattering objects inside the room. These would extend the application of the theory to a wider range of problems.

II. TRANSPORT EQUATION MODELS

In the geometrical acoustic model for room-acoustic predictions, the concept of sound waves is replaced by the notion of sound rays (or phonons\textsuperscript{23}), which significantly reduces the difficulty of directly solving acoustic wave equations. The geometrical acoustic model is an energy-based method, which ignores complex wave phenomena including interference and diffraction. Sound is considered as rays propagating in straight lines with a certain amount of energy. This assumption is considered to be valid in the broad-band high frequency range, where the acoustic wavelength is much smaller than the room dimensions.

Under this circumstance, the sound energy can be described by the concept of a particle distribution function\textsuperscript{19} $\psi(r, \Omega, t)$, where $r$ is the position variable, $\Omega$ is the angular variable, and $t$ is time. More details of these variables are provided in Sec. II A. This particle distribution function can also be termed the sound energy angular flux, which is analogous to the term in neutral particle transport.

A. Three-dimensional transport equation model

This work considers a physical system consisting of a convex area $A$ in the $y,z$-plane, extended in a “tubelike” or “ductlike” manner into the $x$-direction, creating a threedimensional volume $V$ (Fig. 1).

Assuming that $V$ is “long and slender,” implying that if the length $L$ of the duct is finite, then

$$\frac{\sqrt{A}}{L} \ll 1.$$  \hspace{1cm} (1)

Considering sound particle transport within the elongated volume $V$, in which the particle can be reflected specularly or diffusely (Lambert reflection) off the walls, the time-dependent transport equation is\textsuperscript{19}

$$\frac{1}{c} \frac{\partial \psi}{\partial t}(r, \Omega, t) + \Omega \cdot \nabla \psi(r, \Omega, t) + M \psi(r, \Omega, t) = \frac{Q(r, t)}{4\pi},$$  \hspace{0.5cm} (2)

where $\psi$ is the sound angular flux (W m\textsuperscript{-2} sr\textsuperscript{-1}), $c$ is the speed of sound (m s\textsuperscript{-1}), $Q$ is an isotropic sound source term (W m\textsuperscript{-3}), $\Omega$ is the unit vector in direction of particle propagation, and $M$ is the atmospheric attenuation constant (m\textsuperscript{-1}).

The boundary condition for the long side walls is

$$\psi(r, \Omega_0, t) = R \left( 1 - s \right) \psi(r, \hat{\Omega}, t) + \frac{s}{\pi} \int_{\Omega' \cdot n > 0} \Omega' \cdot n \psi(r, \Omega', t) d\Omega',$$  \hspace{0.5cm} (3)

where $R$ is the energy reflection coefficient ($R=1-\alpha$, where $\alpha$ is the absorption coefficient), $s$ is the scattering coefficient, and $n$ is the unit outer normal vector. The first term on the right side of Eq. (3) represents specular reflection, and the second (integral) term describes diffuse reflection. These terms are explained in more detail below. For simplicity in the following, all the walls have the same reflection and scattering coefficients. In addition, it is possible to include sound transmissions from adjacent rooms by adding an external source term.\textsuperscript{24} However, this is beyond the scope of this work and will not be discussed further.

To explain the notation in these equations (Fig. 2), let $i$, $j$, and $k$ denote the unit vector in the direction of the positive $x$-, $y$-, and $z$-axes, respectively, then the position variable (vector):

$$r = (\bar{x}, y, z) = xi + yj + zk.$$  \hspace{1cm} (4)

Also, the unit vector in direction of flight

$$\Omega = (\mu, \sqrt{1 - \mu^2}\cos \gamma, \sqrt{1 - \mu^2}\sin \gamma),$$  \hspace{0.5cm} (5)

where

$$\mu = \cos \theta = \text{polar cosine},$$  \hspace{0.5cm} (6a)

$$\theta = \angle \text{ between } \Omega \text{ and } i,$$

$$\gamma = \angle \text{ between the projection of } \Omega \text{ onto the } j,k\text{-plane and } j = \text{azimuthal angle.}$$  \hspace{0.5cm} (6b)

Equation (5) can be written as

$$\Omega = \mu i + \sqrt{1 - \mu^2}\phi,$$  \hspace{0.5cm} (7)

where $\phi$ is the unit vector

$$\phi = \cos \gamma j + \sin \gamma k.$$  \hspace{0.5cm} (8)

All the notations in Eq. (2) have now been defined.
In Eq. (3), \( \hat{\Omega} \) is the “specular reflection” of the incident direction vector \( \Omega \) across the tangent plane at a point \( r \) on the boundary of \( V \) (Fig. 3), i.e.,

\[
\hat{\Omega} = \Omega - 2(\Omega \cdot n)n.
\]

(9)

Since \( n \) lies in the \( j,k \) plane, \( n \cdot i = 0 \). Equation (9) yields

\[
\hat{\mu} = \hat{\Omega} \cdot i = \Omega \cdot i = \mu,
\]

(10)

and hence \( \hat{\mu} = \mu \). When a particle specularly reflects, its \( \mu \)-value does not change, but its \( \gamma \)-value does change. Equation (9) also implies that

\[
\hat{\Omega} \cdot n = \Omega \cdot n - 2(\Omega \cdot n) = -\Omega \cdot n.
\]

(11)

Thus, the projection of \( \hat{\Omega} \) onto \( n \) is positive, while the projection of \( \Omega \) onto \( n \) is negative and has the same magnitude.

In Eq. (3), \( \Omega' \) is a variable of integration that describes all the unit vectors in the outgoing directions. The integral term and the constant \( 1/\pi \) manifest that the reflected angular flux is uniform in all incoming directions (Lambert’s law).10 Furthermore, the constant \( s \) satisfies \( 0 \leq s \leq 1 \), and \( s \) represents the probability that when a particle scatters off the wall, it scatters diffusely, \( (1-s) \) represents the probability that when a particle scatters off the wall, it scatters specularly.

Moreover, an operation on both sides of Eq. (3) by

\[
\int_{\Omega \cdot n < 0} |\Omega \cdot n| \psi(r,\Omega,t) d\Omega
\]

yields

\[
\int_{\Omega \cdot n < 0} |\Omega \cdot n| \psi(r,\hat{\Omega},t) d\Omega
\]

\[= R \left[ (1-s) \int_{\Omega \cdot n < 0} |\Omega \cdot n| \psi(r,\hat{\Omega},t) d\Omega \right]
\]

\[+ \frac{s}{\pi} \left( \int_{\Omega \cdot n < 0} |\Omega \cdot n| d\Omega \right) \int_{\Omega' \cdot n > 0} \Omega' \cdot n \psi d\Omega'. \]

(13)

However,

\[
\frac{1}{\pi} \int_{\Omega \cdot n < 0} |\Omega \cdot n| d\Omega = \frac{1}{\pi} \int_{-1}^{0} \mu d\mu d\gamma = 1.
\]

(14)

Also, Eq. (11) and \( d\hat{\Omega} = d\Omega \) yield

\[
\int_{\Omega \cdot n < 0} |\Omega \cdot n| \psi(r,\hat{\Omega},t) d\Omega = \int_{\Omega \cdot n > 0} \Omega \cdot n \psi(r,\Omega,t) d\Omega.
\]

(15)

Thus, Eq. (13) becomes

\[
\int_{\Omega \cdot n < 0} |\Omega \cdot n| \psi(r,\hat{\Omega},t) d\Omega = R \int_{\Omega \cdot n > 0} \Omega \cdot n \psi(r,\Omega,t) d\Omega,
\]

(16)

implying that the rate per unit area, at which particles are reflected off the wall at \( r \in \partial V \), equals \( R \) times the rate per unit area, at which particles are incident on the wall at \( r \in \partial V \). Therefore, \( 0 \leq R \leq 1 \) and \( R \) represents the probability that when a particle strikes the wall it will be reflected (specularly or diffusely). This completes the interpretation of the terms in Eqs. (2) and (3).

Finally, for the boundary conditions of the two ends of the volume \( V \), consider two models. The first model only takes specular reflections into account and is simply written as

\[
\psi(0,y,z,\gamma,\mu,t) = R' \psi(0,y,z,\gamma,-\mu,t), \quad 0 < \mu \leq 1,
\]

(17a)

\[
\psi(L,y,z,\gamma,\mu,t) = R'' \psi(L,y,z,\gamma,-\mu,t), \quad -1 \leq \mu < 0,
\]

(17b)

where \( R' \) and \( R'' \) are the reflection coefficients of the two ends, respectively.

For purely diffusely reflecting boundaries, according to Eq. (3),

\[
\psi(r,\Omega,t) = \frac{R'(or R'')}{\pi} \int_{\Omega' \cdot n > 0} \Omega' \cdot n \psi(r,\Omega',t) d\Omega',
\]

\[
\Omega \cdot n < 0,
\]

(18)

which leads to

\[
\psi(0,y,z,\gamma,\mu,t)
\]

\[= \frac{R'}{\pi} \int_{\gamma' = 0}^{2\pi} \int_{\mu' = -1}^{0} (-\mu') \psi(0,y,z,\gamma',\mu',t) d\mu' d\gamma', \quad 0 < \mu \leq 1,
\]

(19a)

\[
\psi(L,y,z,\gamma,\mu,t)
\]

\[= \frac{R''}{\pi} \int_{\gamma' = 0}^{2\pi} \int_{\mu' = 0}^{1} \mu' \psi(L,y,z,\gamma',\mu',t) d\mu' d\gamma', \quad -1 \leq \mu < 0.
\]

(19b)

For partially diffusely reflecting boundary condition, a linear combination of the above two is utilized.

### B. One-dimensional transport approximation

To proceed, define

\[
A' = \int_{A} dydz = \text{cross-sectional area of } A,
\]

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where $ds'$ denotes an increment of arc length.

Combine Eqs. (2) and (7) to get

$$
\frac{1}{c} \frac{\partial \psi}{\partial t}(r, \Omega, t) + \mu \frac{\partial \psi}{\partial x}(r, \Omega, t) + \sqrt{1 - \mu^2} \phi \cdot \nabla \psi(r, \Omega, t) + M \psi(r, \Omega, t) = \frac{Q(r, t)}{4\pi}.
$$

(21)

Intuitively, a very simple way to arrive at a one-dimensional transport equation is to disregard the $y$, $z$-dependency of the distribution function $\psi$ by setting the third term in Eq. (21) to zero. However, this would not give a correct one-dimensional transport equation. As shown below, keeping the third term in Eq. (21) is imperative when integrating Eq. (21) and the boundary condition, i.e., Eq. (3).

An approximation of $\psi$ by the method of weighted residuals is then introduced as

$$
\psi(r, \mu, \gamma; t) = \sum_{j=1}^{N} \alpha_j(y, z, \gamma) \psi_j(x, \mu, t),
$$

(22)

where $\alpha_j$ are specified basis functions and $\psi_j$ are unknown expansion functions. The criterion for selecting the basis functions is that: In a long space, it is reasonable to assume that the sound field is strongly dependent on the long dimension ($x$-axis) and weakly dependent on the lateral coordinates ($y$- and $z$-axes). In other words, it is expected that the sound field varies significantly along the long dimension but insignificantly across the cross section. Similarly, it is assumed that the angle dependence of sound angular flux on $\mu$ is much stronger than on $\gamma$. Therefore, it is decided to separate $y$, $z$, and $\gamma$ from $x$ and $\mu$ by using the basis functions, which are predefined. In this way, the original equation discards the $y$, $z$, and $\gamma$ dependences and is only $x$ and $\mu$ dependent.

Substituting the approximation of $\psi$ generates an error or a residual, which is required to be orthogonal to certain expansions. The criterion for selecting the basis functions is therefore, it is decided to separate $y$, $z$, and $\gamma$ from $x$ and $\mu$ by using the basis functions, which are predefined. In this way, the original equation discards the $y$, $z$, and $\gamma$ dependences and is only $x$ and $\mu$ dependent.

To explicitly show this and determine the equations for the $\psi_j$, an operation on both sides of Eq. (21) by $\int_{\Omega} \int_{\phi_n < 0} \beta_j(r) \gamma(\cdot) d\gamma dy dz$ and an operation on both sides of Eq. (3) by $\int_{\Omega} \int_{\phi_n < 0} \beta_j(r) \gamma(\cdot) d\gamma dy dz$ yield

$$
\int_{\Omega} \int_{\phi_n < 0} \left[ \frac{1}{c} \frac{\partial \psi}{\partial t} + \mu \frac{\partial \psi}{\partial x} + \sqrt{1 - \mu^2} \phi \cdot \nabla \psi + M \psi \right] d\gamma dy dz = \frac{Q(r, t)}{4\pi},
$$

(23)

$$
\int_{\Omega} \int_{\phi_n < 0} \beta_j \left[ \psi - R \left( 1 - s \right) \psi \right] dy dz + \frac{\pi}{\Omega} \int_{\phi_n > 0} \Omega' \cdot \nabla \psi d\Omega' = 0.
$$

(24)

In addition, $\alpha_j$ and $\beta_j$ are assumed to satisfy

$$
\frac{1}{2\pi A} \int_{A} \int_{0}^{2\pi} \alpha_j \beta_i d\gamma dy dz = \delta_{ij}, \quad 1 \leq i, \quad j \leq N,
$$

(25)

where $\delta_{ij}$ is the Kronecker delta. Equations (22) and (25) imply that

$$
\psi_i = \frac{1}{2\pi A} \int_{A} \int_{0}^{2\pi} \beta_i \psi d\gamma dy dz, \quad 1 \leq i \leq N.
$$

(26)

For a point source at $r_0 = (x_0, y_0, z_0)$, i.e., $Q(r, t) = Q(t) \delta(x-x_0) \delta(y-y_0) \delta(z-z_0)$, apply the divergence theorem to Eq. (23), then use Eq. (24), and finally arrive at the following set of $N$ transport equations (see Appendix for details):

$$
\frac{1}{c} \frac{\partial \psi_i}{\partial t} + \mu \frac{\partial \psi_i}{\partial x} + M \psi_i + \sqrt{1 - \mu^2} \sum_{j=1}^{N} a_{ij} \psi_j = \frac{2Rs}{\pi} \sqrt{1 - \mu^2} \int_{-1}^{1} \psi_j(\mu') d\mu' + Q_i,
$$

(27)

where the derivation of $a_{ij}$ and $b_{ij}$ can be found in the Appendix. For example,

$$
a_{ij} = \frac{1 - R(1 - s)}{2\pi A} \left[ \int_{\phi_n > 0} \phi \cdot n \beta_i \psi_j d\gamma \right], \quad i = 1
$$

(28)

$$
b_{ij} = \frac{1}{4\pi A} \left[ \left( \int_{\phi_n < 0} \phi \cdot n \beta_j d\gamma \right) \times \left( \int_{\phi_n > 0} \phi \cdot n \alpha_i d\gamma \right) \int_{-1}^{1} dy dz \right], \quad 1 \leq i \leq 2,
$$

(29)

and

$$
Q_i = \frac{1}{2\pi A} \int_{A} \int_{0}^{2\pi} \beta_i Q(t) \delta(r-r_0) d\gamma dy dz.
$$

(30)

In addition, substitution of Eqs. (17a) and (17b) into Eq. (26) yields the purely specular boundary condition for the two ends as

$$
\psi_i(0, \mu, t) = R' \psi_i(0, -\mu, t), \quad 0 < \mu \leq 1,
$$

(31a)

$$
\psi_i(L, \mu, t) = R'' \psi_i(L, -\mu, t), \quad -1 \leq \mu < 0.
$$

(31b)

Similarly, for diffuse reflection,

$$
\psi_i(0, \mu, t) = \frac{R'}{\pi} \times \frac{1}{2\pi A} \int_{A} \int_{0}^{2\pi} \beta_j \left[ \int_{-1}^{1} (1 - \mu') ^{2} \psi(0, y, z, \mu', \gamma', t) d\mu' d\gamma' \right] d\gamma dy dz,
$$

(32a)

$$
0 < \mu \leq 1,
$$

(32b)
Now the exact three-dimensional equations, i.e., Eqs. (2) and (3), have been reduced to a coupled set of $N$ one-dimensional transport equations, in the expectation that the resulting computation will be less cumbersome. In this set of equations, $\alpha_j$ and $\beta_i$ have to be specified. Large $N$ should increase the computational load but should also reduce the error of the approximation. Thus, the choice of $N$, the number of transport equations in the approximate one-dimensional model, is a compromise between computational effort and accuracy. This study only considers $N=1$ and 2, which are treated in detail below.

1. One-group model ($N=1$)

This subsection begins with the choice of the basis functions and weight functions. There are different ways to select these functions, for example, the collocation method (the weight functions are chosen to be Dirac delta functions), the least-squares method (which uses derivatives of the residual itself as weight functions), and the Galerkin method. The Galerkin method is preferred because it has already been shown to be accurate for this specific transport equation model. In the Galerkin method, the weight functions are chosen to be identical to the basis functions, and for $N=1$

$$\alpha_1(y,z,\gamma,t) = \beta_1(y,z,\gamma,t) = 1.$$  \hspace{1cm} (33)

The constant of 1 is a convenient choice suggested in previous work. Thus

$$\psi(r,\mu,\gamma,t) = \psi_1(x,\mu,t).$$  \hspace{1cm} (34)

Based on this equation, the three-dimensional transport equation is reduced to a truly “one-dimensional model” since the $y$ and $z$ coordinates disappear. It indicates that the $y$, $z$, and $\gamma$-dependences of $\psi$ are weak; i.e., the energy is almost uniform in the $y$-$z$ plane and over the $\gamma$ angle. This is expected to be a good approximation when the absorption on the boundary is weak and the long space is sufficiently narrow. The theoretical demonstration of this statement has been given in Ref. 22, which is conveyed in a rather mathematical way. It might be more illustrating to explain this from the acoustic point of view: if the absorption on the side walls are strong, the variation in the sound energy across the cross section will be significant, which has been shown in previous literature, e.g., Ref. 12. This implies that the sound energy is no more weakly $y$- and $z$-dependent, which violates the assumption of the one-group model. It has also been theoretically proved that this approximation is valid if the receiver is sufficiently far away from the ends of the long space. Substitution of Eq. (33) into Eqs. (28) and (29) yields

$$\psi_1(L,\mu,t) = \frac{R''}{\pi} \times \frac{1}{2\pi A'} \int_{A}^{2\pi} \int_{0}^{2\pi} \int_{0}^{1} \mu' \psi(0,y,z,\mu',t) d\mu' d\gamma d\gamma' \int_{0}^{1} d\gamma dy dz, \quad -1 \leq \mu < 0,$$

\hspace{1cm} (32b)

The one-dimensional transport equation model for $N=1$ is then

$$\frac{1}{c} \frac{\partial \psi_1}{\partial t} + \mu \frac{\partial \psi_1}{\partial x} + \left[ M + \sqrt{1-\mu^2} \frac{1}{\lambda} \right] \psi_1 = \frac{2Rs}{\lambda \pi} \int_{-1}^{1} \sqrt{1-\mu'^2} \psi_1(\mu')d\mu',$$ \hspace{1cm} (36)

where

$$\lambda = \frac{\pi A'}{L'}.$$  \hspace{1cm} (37)

The specular boundary conditions for the two ends are

$$\psi_1(0,\mu,t) = R' \psi_1(0,-\mu,t), \quad 0 < \mu \leq 1,$$

$$\psi_1(L,\mu,t) = R' \psi_1(L,-\mu,t), \quad -1 \leq \mu < 0.$$ \hspace{1cm} (38a)

Also, the diffuse boundary conditions are

$$\psi_1(0,\mu,t) = 2R' \int_{-1}^{0} (\mu') \psi_1(0,\mu',t)d\mu', \quad 0 < \mu \leq 1,$$

$$\psi_1(L,\mu,t) = 2R' \int_{0}^{1} \mu' \psi_1(L,\mu',t)d\mu', \quad -1 \leq \mu < 0.$$ \hspace{1cm} (39a)

This one-group one-dimensional transport equation is the same for two different structures having the same ratio of $A'$ to $L'$. To explain the physical interpretation of the constant $\lambda$ in Eq. (37), a function $D(y,z,\phi)$ is defined as the distance from a point $(x,y,z)$ in the interior of the volume to the inner wall in the direction $-\phi$ so that

$$\phi \cdot \nabla D(y,z,\phi) = 1 \quad (y,z) \in A,$$

$$D(y,z,\phi) = 0 \quad (y,z) \in \partial A, \quad \phi \cdot n < 0,$$ \hspace{1cm} (40a)

where $n$ is the unit outer normal vector. If $r_0=(x,y,z) \in \partial A$ and $\phi \cdot n < 0$ (i.e., $\phi$ points into $A$), then on the line $r(S)=r_0+S\Omega$, Eq. (40) yields $D[r(S),\phi]=S$. (Fig. 4).
At the point \( r_1 \in \partial A \), where the line “exits” \( A \), \( D(r_1, \phi) = \|r_1 - r_0\| = \) distance through \( A \) in the direction \( \phi = l(r_1, \phi) \) (Fig. 5).

Now an operation on both sides of Eq. (40a) by \( \int_{A} \int_{\gamma=0}^{2\pi} d\gamma dydz \) yields

\[
2\pi A = \int_{A} \int_{\gamma=0}^{2\pi} \phi \cdot \nabla D d\gamma dydz \\
= \int_{\gamma=0}^{2\pi} \left( \int_{\partial A} n \cdot \phi D ds' \right) d\gamma \\
= \int_{\partial A} \left( \int_{\phi n > 0} n \cdot \phi D d\gamma \right) ds' \\
= \int_{\partial A} \int_{\phi n > 0} n \cdot \phi l(r, \phi) d\gamma ds'.
\]

But also,

\[
\int_{\partial A} \left( \int_{\phi n > 0} \phi n d\gamma \right) ds' = \int_{\partial A} 2ds' = 2L'.
\]

Hence, \( \lambda \) has the geometrical interpretation:

\[
\lambda = \langle l \rangle = \frac{\int_{\partial A} \int_{\phi n > 0} n \cdot \phi l(r, \phi) d\gamma ds'}{\int_{\partial A} \int_{\phi n > 0} n \cdot \phi ds'} = \frac{\pi A'}{L'},
\]

which is the classical diffuse mean free path length for two dimensions, or mean chord length across \( A \). This completes the discussion of the one-group transport equation model.

### 2. Two-group model (N=2)

The choice of the basis functions and weight functions for the two-group model (N=2) is based on a linear combination of the constant function 1, and \( D(y, z, \gamma) \), defined in Sec. II B 1 to be the distance from a point \((x, y, z)\) in the interior of the volume to the inner wall in the direction \(-\phi\). They are expressed explicitly as

\[
\alpha_1(y, z, \gamma, t) = \beta_1(y, z, \gamma, t) = 1,
\]

\[
\alpha_2(y, z, \gamma, t) = \beta_2(y, z, \gamma, t) = u[D(y, z, \phi) - v],
\]

where

\[
u = \frac{1}{2\pi A'} \int_{0}^{2\pi} D(y, z, \phi) d\gamma dydz,
\]

or in the form of curve integral:

\[
u = \frac{1}{4\pi A'} \int_{\phi n > 0} \phi \cdot n D d\gamma dydz',
\]

which can be obtained using the divergence theorem (Gauss’s theorem) and the identity

\[
D = (\phi \cdot \nabla D^2)/2.
\]

To show how to derive the matrices \( a_{ij} \) and \( b_{ij} \) using \( a_{12} \) as an example:

\[
a_{12} = \frac{1-R(1-s)}{2\pi A'} \int_{\partial A} \int_{\phi n > 0} \phi \cdot n u(D - v)d\gamma ds' \\
= \frac{1-R(1-s)}{2\pi A'} \left[ \int_{\partial A} \int_{\phi n > 0} \phi \cdot n D d\gamma ds' \\
- \int_{\partial A} \int_{\phi n > 0} \phi \cdot n D d\gamma ds' \right] \\
= \frac{1-R(1-s)}{2\pi A'} \left[ \int_{\partial A} \int_{\phi n > 0} \phi \cdot \nabla D d\gamma dydz - v2L' \right] \\
= \left[ 1-R(1-s) \right] \left( u - uL'/(\pi A') \right).
\]

In similar fashion, other matrix elements can be derived. Omitting the lengthy algebra, the final results of the matrices \( a_{ij} \) and \( b_{ij} \) can be expressed as

\[
a_{11} = [1-R(1-s)] \frac{L'}{\pi A'},
\]

\[
a_{12} = [1-R(1-s)] \left( u - uL'/(\pi A') \right),
\]

\[
a_{21} = \frac{(1-s)uL'}{\pi A'},
\]

\[
a_{22} = \frac{uvL'}{\pi A'} + R(1-s)uv \left( u - \frac{uL'}{\pi A'} \right),
\]

\[
b_{ij} = \left[ \frac{L'/(\pi A')}{u - uL'/(\pi A')} \right] - \left[ \frac{uL'/(\pi A')}{u - uL'/(\pi A')} \right].
\]

The specular boundary conditions on the two ends are

\[
\psi_1(0, \mu, t) = R' \psi_1(0, -\mu, t), \quad 0 < \mu \leq 1,
\]

where

\[

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\[ \psi_1(L, \mu, t) = R^\mu \psi_1(L, -\mu, t), \quad -1 \leq \mu < 0, \quad (54b) \]
\[ \psi_2(0, \mu, t) = R^\mu \psi_2(0, -\mu, t), \quad 0 < \mu \leq 1, \quad (54c) \]
\[ \psi_2(L, \mu, t) = R^\mu \psi_2(L, -\mu, t), \quad -1 \leq \mu < 0. \quad (54d) \]

The diffuse boundary conditions are given by
\[ \psi_1(0, \mu, t) = 2R^\mu \int_{-1}^{0} (-\mu') \psi_1(0, \mu', t) d\mu', \quad 0 < \mu \leq 1, \quad (55a) \]
\[ \psi_1(L, \mu, t) = 2R^\mu \int_{0}^{1} \mu' \psi_1(0, \mu', t) d\mu', \quad -1 \leq \mu < 0, \quad (55b) \]

\[ \psi_2(0, \mu, t) = \frac{R^\mu u^2}{2\pi A'} \left[ \int_{A} \left( \int_{0}^{2\pi} D d\gamma \right)^2 d\gamma d^2 - 4\pi^2 v^2 A' \right] \times \int_{-1}^{0} -\mu' \psi_2(0, \mu', t) d\mu', \quad 0 < \mu \leq 1, \quad (55c) \]
\[ \psi_2(L, \mu, t) = \frac{R^\mu u^2}{2\pi A'} \left[ \int_{A} \left( \int_{0}^{2\pi} D d\gamma \right)^2 d\gamma d^2 - 4\pi^2 v^2 A' \right] \times \int_{0}^{1} \mu' \psi_2(0, \mu', t) d\mu', \quad -1 \leq \mu < 0, \quad (55d) \]

where for a long space with circular cross section \( A, \) \( u, \) and \( v \) are given analytically as
\[ u = 3\pi(9\pi^2 - 64)^{-1/2}/\rho' \quad \text{and} \quad v = 8\rho'/(3\pi), \quad (56) \]
with \( \rho' \) being the radius of the circle. \( a_{ij} \) and \( b_{ij} \) are then immediately known by employing Eqs. (52) and (53). For noncircular geometries, \( u \) and \( v \) can either be analytically or numerically derived. For instance, implementing Eqs. (48) and (49) for a square of width of \( d \) yields
\[ u = 3.25/d \quad \text{and} \quad v = 0.47d. \quad (57) \]

Comparing the one-group and two-group models, the latter has an additional transport equation, which increases its complexity. However, the accuracy should also be increased, since the energy is not assumed to be uniform in the \( y-z \) plane or over the \( \gamma \) angle as the one-group model does. While the one-group model is legitimate when absorption is weak, the two-group model should be valid for a wider range of absorption coefficients. In addition, the function \( D \) distinguishes different types of cross-sectional geometry by generating different \( u \) and \( v, \) and it brings back the coordinates \( y, \) \( z, \) and \( \gamma. \) Thus the two-group model allows the consideration of the positions of both a receiver and source in the \( y-z \) plane.

Next, consider a simple numerical example to compare the accuracy of the one-group and two-group models. Assuming a semi-infinite circular duct with a radius of 1 m, and having all the surfaces perfectly absorbing, the sound energy decay along the long dimension is investigated when an omnidirectional point source is located at the center of the origin \((x = y = z = 0). \) This problem is equivalent to the sound propagation from a point source to a free space, where the sound field can be analytically solved. Therefore, we can use the exact analytic solution as the benchmark.

By assigning the reflection coefficient \( R \) and the attenuation coefficient \( M \) to zero in the one-group model, the analytic solution can be obtained as
\[ \psi_1(x, \mu) = \frac{Q}{4\pi A \mu} \exp \left( -\frac{\sqrt{1-\mu^2}}{\lambda \mu} \right), \quad (58) \]

where \( Q \) is the sound source power, and the time \( t \) is discarded Since the steady-state solution is of concern in this case. For the two-group model, a numerical solution is obtained. A follow-up paper \( 28 \) will report on details of the numerical implementations.

So far only the angular flux \( \psi \) has been discussed. However, in room acoustics, the sound energy or the sound pressure level is most frequently focused. Knowing the angular flux \( \psi, \) the sound pressure level can be written as
\[ L_p(r) = 10 \log \left( \frac{I(r) \rho c}{P_{\text{ref}}} \right), \quad (59) \]

where
\[ I = \int_{0}^{2\pi} \int_{-1}^{1} \psi d\mu d\gamma \quad (60) \]
is the magnitude of sound intensity, \( \rho \) is the air density, and \( P_{\text{ref}} = 2 \times 10^{-5} \) Pa is the pressure reference. Thus, the intensity is an integral of the angular flux over all the angles. While the Monte Carlo simulation normally only provides the sound energy, which is angularly independent, the transport equation model provides more detailed angular information through the angular flux \( \psi. \)

By numerically implementing Eq. (60) for the solutions of the one-group and two-group models, the sound intensities along the long dimension can be obtained and are plotted in Fig. 6. These solutions are normalized at a distance of 1 m to the source. The exact solution \( \psi(r, \Omega) \) of Eq. (2) with \( \partial \psi/\partial t = 0, M = 0, \) and \( Q(r) = Q \partial \Omega \) satisfies
\[ \int_{0}^{2\pi} \int_{-1}^{1} \psi(r, \Omega) d\mu d\gamma = \frac{Q}{4\pi r^2}, \quad (61) \]
where \( r = \sqrt{x^2 + y^2 + z^2} \) is the distance from the source. Figure 6 illustrates this result, which is normalized at \( x = 1, \) as a benchmark. Figure 6(a) indicates that the two-group solution decays at a faster and more accurate rate than the one-group solution. It seems plausible that the one-group model is not such a bad match to the analytic solution at a long distance. However, Fig. 6(b), illustrating the sound intensity on a logarithmic scale, indicates that the one-group model becomes even worse at a long distance. Therefore, the one-group model is not able to accurately predict the sound field in this open-space case. This is because, as demonstrated in Sec. II B 1, the one-group model will fail when the absorption on
the wall is strong (as in this case, where the absorption coefficient is 1.0). Additional simulations show that the numerical results of both one- and two-group models converge to the analytic solution when narrowing the duct.

Indeed, the sound propagation in a free space is somehow an unrealistic case. It is expected that the result would be totally different if the tube wall had been given a reflection coefficient close to 1, especially for the one-group model which is validated when the reflection is strong. However, the purpose of this example is only to demonstrate that the two-group model is more accurate than the one-group model. And the reason for choosing this simple example is that the analytic solutions of the one-group model as well as the three-dimensional exact equation can be easily found. More complicated cases (reflection coefficient being non-zero) require sophisticated numerical methods for both one- and two-group models, which will be detailed in a follow-up paper.28

C. Other considerations

1. Sources

This subsection briefly discusses the possibility of taking into account different types of sound sources and source directivity. This demonstrates the flexibility of the present model.

So far, only the omnidirectional point source has been considered. According to Eq. (30) and the definition of the Dirac delta function, for the two-group model,

\[ Q_1 = \frac{Q(t)}{4\pi A} \delta(x - x_0), \]  

\[ Q_2 = \frac{Q(t)}{8\pi A} \delta(x - x_0) \times \int_0^{2\pi} u[D(y_0, \phi)] d\phi - v] d\gamma. \]  

(62b)

For a point source at \( r_0 \) with directivity, the source term in Eq. (2) is written as \( Q(\gamma, \mu, t) \delta(r - r_0) \) instead, and similarly

\[ Q_1 = \frac{1}{2\pi A} \int_0^{2\pi} Q(\gamma, \mu, t) d\gamma, \]  

\[ Q_2 = \frac{1}{2\pi A} \int_0^{2\pi} u[D(y_0, \phi)] - v] Q(\gamma, \mu, t) d\gamma. \]  

(63a)

(63b)

Besides the point source, which is the most effective source in room-acoustic simulations, line sources and plane sources can also be treated by the one-dimensional transport equation model. For a line source at \( x = x_0, y = y_0 \) which spans from \( z_0 \) to \( z_1 \), the source term is written as \( Q(\gamma, \mu, t, \delta(x - x_0) \delta(y - y_0) \chi_{[z_0, z_1]}(z), \) where \( \chi \) is the indicator function,

\[ \chi_B(x) = 1 \quad \text{if} \quad x \in B, \]  

\[ \chi_B(x) = 0 \quad \text{if} \quad x \notin B. \]  

(64)

\( Q_1 \) immediately becomes

\[ Q_1 = \frac{1}{2\pi A} \int_0^{2\pi} Q(\gamma, \mu, t) d\gamma, \]  

\[ Q_2 = \frac{1}{2\pi A} \int_0^{2\pi} u[D(y_0, \phi)] - v] d\gamma. \]  

(65a)

(65b)

In the same fashion, the source term for a plane source at \( x = x_0 \) is \( Q(\gamma, \mu, t) \delta(x - x_0) \). \( Q_i \) are written as

\[ Q_1 = \frac{1}{2\pi} \int_0^{2\pi} Q(\gamma, \mu, t) d\gamma, \]  

\[ Q_2 = \frac{1}{2\pi A} \int_0^{2\pi} Q(\gamma, \mu, t) \int_{z_0}^{z_1} u[D(y_0, z, \phi) - v] dz d\gamma. \]  

(66a)

(66b)

For one-group model, only \( Q_1 \) is needed.

2. Furnished rooms

In furnished rooms (sometimes also called fitted rooms), the interior contains noticeable objects (fittings), e.g., machines, chairs, and desks. Some factories, classrooms, and offices are studied as furnished rooms. In these cases, the theory for empty rooms no longer holds—the fittings inside the room need to be considered. In general, the sound field in a furnished room is complex due to different locations, absorption, and scattering coefficients of objects inside the room. Our goal is to model a simple case where the objects inside the room have more or less the same absorption and scattering coefficients, and the positions of
these scattering objects can be statistically described by the mean free path between them, which can be obtained by a best-fit approach.\(^{31}\) To further simplify the problem, the scattering of the sound by the fittings is modeled as a uniform isotropic scattering. For furnished rooms, the three-dimensional transport equation is
\[
\frac{1}{c} \frac{\partial \psi}{\partial t} (r, \Omega, t) + \Omega \cdot \nabla \psi (r, \Omega, t) + (M + \sigma_T) \psi (r, \Omega, t) = \int_{\Omega} Q (r, \Omega, t) d\Omega',
\]
where \(\sigma_T = \sigma_s + \sigma_a\), and \(\sigma_s\) and \(\sigma_a\) are the so-called scattering and absorption coefficients for the fittings, respectively. It can be shown that\(^{13}\)
\[
\sigma_T = \frac{1}{\lambda'}, \quad \sigma_A = \frac{-\ln(1 - \alpha)}{\lambda'},
\]
\(\lambda'\) is the mean free path between the scattering objects in the room, and \(\alpha\) is the sound energy absorption coefficient of the fittings. A similar equation is documented in Ref. 32, with the absorption term being
\[
\sigma_A = \frac{\alpha}{\lambda'}.
\]
This difference has been explained in Refs. 13 and 33.

Repeating the same analysis for the empty room case, the one-dimensional transport equation model for the furnished rooms can be obtained. For example, the one-group model is written as
\[
\frac{1}{c} \frac{\partial \psi_1}{\partial t} + \mu \frac{\partial \psi_1}{\partial x} + (M + \sigma_T) \psi_1 + \sqrt{1 - \mu^2} a_{11} \psi_1 = \int_{\Omega} Q_1 (r, \Omega, t) d\Omega',
\]
where \(a_{11}\) is the sound energy absorption coefficient of the fittings. A similar equation is documented in Ref. 32, with the absorption term being
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\[
\sigma_A = \frac{\alpha}{\lambda'}.
\]
This difference has been explained in Refs. 13 and 33.
\[ F_i = -\frac{1 - \mu_i^2}{2\pi A^2} \left\{ \int_{\Omega^+} \int_{\phi n > 0} \phi \cdot n \beta_i \psi d\gamma d\gamma' \right\} + \int_{\Omega^+} \int_{\phi n > 0} \phi \cdot n \beta_i R \left[ (1 - s) \psi(\Omega') \right] d\gamma d\gamma' + \frac{1}{\pi} \int_{\Omega^+} \phi \cdot n R \left[ \left( 1 - \mu_i^2 \right) \psi d\mu' d\gamma' \right] d\gamma + \int_{\Omega^+} \psi \phi \cdot \nabla \beta_i d\gamma d\gamma' \right\}. \]  

From Eqs. (10), (11), and (40b),

\[ \int_{\phi n < 0} \phi \cdot n \psi(\Omega) d\gamma = -\int_{\phi n > 0} \phi \cdot n \psi(\Omega) d\gamma = -\int_{\phi n > 0} \phi \cdot n \psi(\Omega) d\gamma. \]  

and

\[ \int_{\phi n < 0} \phi \cdot n u(D - v) \psi(\Omega) d\gamma = 0 - uv \int_{\phi n < 0} \phi \cdot n \psi(\Omega) d\gamma = uv \int_{\phi n > 0} \phi \cdot n \psi(\Omega) d\gamma. \]  

Substitution of Eqs. (44), (45), (A5), and (A6) into Eq. (A4) yields

\[ F_i = -\frac{1 - \mu_i^2}{2\pi A^2} \left\{ \left[ 1 - R(1 - s) \right] \int_{\Omega^+} \int_{\phi n > 0} \phi \cdot n \psi d\gamma d\gamma' \right\} + \frac{R s}{\pi} \int_{\Omega^+} \int_{\phi n < 0} \phi \cdot n \Omega' \cdot n \psi(\Omega') d\Omega' d\gamma d\gamma' \]  

and

\[ F_i = -\frac{1 - \mu_i^2}{2\pi A^2} \left\{ \int_{\Omega^+} \int_{\phi n > 0} \phi \cdot n \beta_i \psi d\gamma d\gamma' + R(1 - s)uv \right\} \times \left[ \int_{\Omega^+} \int_{\phi n > 0} \phi \cdot n \psi d\gamma d\gamma' + \frac{R s}{\pi} \int_{\Omega^+} \int_{\phi n < 0} \phi \cdot n \beta_i d\gamma d\gamma' \right] \]  

For \( F_i \), where \( i > 2 \), similar equations can be derived if \( \alpha_i \) and \( \beta_i \) are known. This is, however, not discussed here. Moreover, by recognizing that

\[ \int_{\Omega^+} \Omega' \cdot n \psi(\Omega') d\Omega' d\gamma d\gamma' = \int_{\phi n > 0} \phi \cdot n \int_{\phi n > 0} \left[ 1 - \mu_i^2 \right] \psi d\mu' \]  

and employing Eq. (22),

\[ \int_{\phi n < 0} \phi \cdot n \beta_i \left[ \int_{\Omega^+} \Omega' \cdot n \psi(\Omega') d\Omega' \right] d\gamma = \int_{\phi n < 0} \phi \cdot n \beta_i \left[ \int_{\phi n > 0} \Omega' \cdot n \psi(\Omega') d\Omega' \right] d\gamma \]  

Substituting Eq. (A10) into Eqs. (A7) and (A8), the one-dimensional transport model, Eq. (27), is finally obtained.