8.7 Exercises

Horns and Ducts

Problem 8.1. Use Euler’s equation in (8.1) and the modified continuity equation in (8.6), show that the general Webster’s equation can be expressed as

\[
\frac{\partial^2 p}{\partial x^2} + \left[ \frac{1}{A(x)} \frac{dA}{dx} \right] \frac{\partial p}{\partial x} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}.
\]

(8.61)

Problem 8.2. Show that equation (8.11)

\[
\frac{\partial^2 (p x)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 (p x)}{\partial t^2}
\]

(8.62)

can be expressed as

\[
\frac{\partial^2 p}{\partial x^2} + \frac{2}{x} \frac{\partial p}{\partial x} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2},
\]

(8.63)
as stated in equation (8.9)

Problem 8.3. A horn radiator is made of an electrodynamic loudspeaker, a pressure chamber and an exponential horn – Fig. 8.12. The horn is assumed long enough, so that reflected waves from the opening end can be ignored. Furthermore, the pressure chamber is very small in comparison with the shortest wave lengths under consideration.

(a) Determine the mechanic (input) impedance of the horn for \( \omega \gg \omega_l \). The horn is excited by a constant velocity. Find the horn’s flare coefficient so that the radiated power, above a frequency of \( f_u = 500 \text{Hz} \), does not vary by more than 3 dB.
(b) Develop an equivalent electric circuit for the pressure chamber

(c) Develop an equivalent circuit for the entire system with $\omega >> \omega_l$. Simplify the equivalent circuit for the frequency range where frequency course remains constant

(d) Determine the transformation coefficient (area ratio) of the pressure chamber so that the horn radiates the maximum sound power, whereby

- Specific impedance of air \( \ldots \rho c = 406 \text{Ns/m}^3 \)
- Input resistance of the loudspeaker \( \ldots R_1 = 24 \Omega \)
- Inner resistance of the signal generator \( \ldots R_g = 24 \Omega \)
- Square value of the transducer coefficient \( \ldots B^2 l^2 = 44 \text{Vs/m}^2 \)
- Mechanic resistance \( \ldots r = 1.2 \text{Ns/m} \)
- Diameter of the speaker membrane \( \ldots d = 6 \text{cm} \)

**Problem 8.4.** Given an arrangement as sketched in Fig. 8.13.

![Fig. 8.13. Stepped tube composed of two sections with pressure chamber](image)

(a) Determine the function \( H(\omega) = \frac{q_m(\omega)}{q_g(\omega)} \)

(b) Find the poles of this function. Note that the termination impedance is assumed very small in comparison to the tube resistance (tube \( l_2 \) is open at the right end)
Problem 8.5. When a (plane) wave traveling towards a rigidly terminated tube, what is the field impedance $Z_s$ at the surface indicated by the dot-line as sketched in Fig. 8.14?

Fig. 8.14. Air layer of thickness $D$ in front of rigidly terminated tube. The wave length is much larger than the tube diameter.
16.8 Solutions for Chapter 8

Problem 8.1

Focus Given the Euler equation:
\[-\frac{\partial p}{\partial x} = \frac{\partial v}{\partial t}, \quad (16.192)\]
and the modified continuity equation:
\[-\left( \frac{\partial v}{\partial x} + \frac{1}{A(x)} \frac{dA}{dx} v \right) = \kappa \frac{\partial p}{\partial t}. \quad (16.193)\]

Task Derive Webster’s equation:
\[\frac{\partial^2 p}{\partial x^2} + \left[ \frac{1}{A(x)} \frac{dA}{dx} \right] \frac{\partial p}{\partial x} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}. \quad (16.194)\]

Setup

Note:
\[\kappa = \frac{1}{c^2 \rho}. \quad (16.195)\]

Solve

First, the Euler equation is rearranged by differentiating on both sides with respect of \(x\) such that:
\[-\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial v}{\partial t}, \quad -\frac{1}{\rho} \frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial t}. \quad (16.196)\]

Then, take the derivative with respect to time \(t\) on both sides of the modified continuity equation:
\[-\left( \frac{\partial^2 v}{\partial x \partial t} + \frac{1}{A(x)} \frac{dA}{dx} \frac{\partial v}{\partial t} \right) = \kappa \frac{\partial^2 p}{\partial t^2}. \quad (16.197)\]
so substitution of the rearranged Euler equation in (16.196) into the modified continuity equation yields:
\[- \left[ \left( -\frac{1}{\rho} \frac{\partial^2 p}{\partial x^2} \right) + \frac{1}{A(x)} \frac{dA}{dx} \left( -\frac{1}{\rho} \frac{\partial p}{\partial x} \right) \right] = \kappa \frac{\partial^2 p}{\partial t^2}, \quad (16.198)\]
so expanding \(\kappa\) in eq.(16.195) and multiplying both sides by \(\rho\)
\[\frac{\partial^2 p}{\partial x^2} + \left[ \frac{1}{A(x)} \frac{dA}{dx} \right] \frac{\partial p}{\partial x} = \frac{\rho}{c^2 \rho} \frac{\partial^2 p}{\partial t^2}, \quad (16.199)\]
which reduces to:
\[\frac{\partial^2 p}{\partial x^2} + \left[ \frac{1}{A(x)} \frac{dA}{dx} \right] \frac{\partial p}{\partial x} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}. \quad (16.200)\]
Problem 8.2

Task

Show that:

\[
\frac{\partial^2(px)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2(px)}{\partial t^2}
\]  
(16.201)

can be expressed as

\[
\frac{\partial^2p}{\partial x^2} + \frac{2}{x} \frac{\partial p}{\partial x} = \frac{1}{c^2} \frac{\partial^2p}{\partial t^2}.
\]  
(16.202)

Setup

The product rule is defined as:

\[
(f \cdot g)' = f'g + fg'.
\]  
(16.203)

The Euler equation:

\[- \frac{\partial p}{\partial x} = \rho \frac{\partial v}{\partial t}.
\]  
(16.204)

Solve

Taking the second derivative of the left-hand side of the equation:

\[
\frac{\partial}{\partial x} \left[ \frac{\partial(px)}{\partial x} \right] = \frac{\partial}{\partial x} \left[ \frac{\partial p}{\partial x} x + p \right] = \frac{\partial^2p}{\partial x^2} x + \frac{\partial p}{\partial x} + \frac{\partial p}{\partial x} = \frac{\partial^2p}{\partial x^2} x + 2 \frac{\partial p}{\partial x},
\]  
(16.205)

while on the right-hand side:

\[
\frac{1}{c^2} \frac{\partial}{\partial t} \left[ \frac{\partial(px)}{\partial t} \right] = \frac{1}{c^2} \frac{\partial}{\partial t} \left[ \frac{\partial p}{\partial x} x + p(0) \right] = \frac{1}{c^2} \frac{\partial^2p}{\partial t^2} x.
\]  
(16.206)

Substitution of these results into the original equation results in

\[
\frac{\partial^2p}{\partial x^2} x + 2 \frac{\partial p}{\partial x} = \frac{1}{c^2} \frac{\partial^2p}{\partial t^2} x.
\]  
(16.207)

Dividing both sides by x:

\[
\frac{\partial^2p}{\partial x^2} + \frac{2}{x} \frac{\partial p}{\partial x} = \frac{1}{c^2} \frac{\partial^2p}{\partial t^2}.
\]  
(16.208)

Problem 8.3

A horn radiator is made of an electrodynamic loudspeaker, a pressure chamber and an exponential horn as shown in Fig. 15.14. The horn is assumed long enough, so that reflected waves from the opening end can be ignored. Furthermore, the pressure chamber is very small in comparison with the shortest wave lengths under consideration.
Task

(a) (1) Determine the mechanic input impedance of the horn for $\omega >> \omega_l$. The horn be excited by a constant velocity. (2) Find the horn’s flare coefficient $\varepsilon$ so that the radiated power, above a frequency of $f_h = 500$ Hz, does not vary by more than 3 dB.

Setup

(a.1) The field impedance of an exponential horn is

$$Z_f = \varrho c \left[ \sqrt{1 - \left(\frac{\omega_l}{\omega}\right)^2} + j \left(\frac{\omega_l}{\omega}\right) \right].$$  \hspace{1cm} (16.209)

Solve

Given the mouth cross-sectional area $A_0$ of the horn, the input mechanic impedance becomes

$$Z_{mech} = A_0 \varrho c \left[ \sqrt{1 - \left(\frac{\omega_l}{\omega}\right)^2} + j \left(\frac{\omega_l}{\omega}\right) \right].$$  \hspace{1cm} (16.210)

For the frequency $\omega >> \omega_l$, the term $\omega_l/\omega$ inside eq.(16.210) becomes negligible, the field impedance of the horn is approaching to

$$Z_{mech} = A_0 \varrho c.$$  \hspace{1cm} (16.211)

Setup

(a.2) The radiated power of the exponential horn is

$$\bar{P} = \frac{1}{2} \text{Re}\{Z_{rad}\} |\varphi|^2,$$  \hspace{1cm} (16.212)

where the real part of the radiation impedance

$$\text{Re}\{Z_{rad}\} = A_0 \varrho c \sqrt{1 - \left(\frac{\omega_l}{\omega}\right)^2}.$$  \hspace{1cm} (16.213)

Solve

For the frequency $\omega >> \omega_l$, the radiation power becomes

$$\bar{P}_\infty = \frac{1}{2} A_0 \varrho c |\varphi|^2,$$  \hspace{1cm} (16.214)

where the velocity $\varphi$ is a constant. For the frequency $f = f_h = f_{500Hz} = 500$ Hz, the radiated power,

$$\bar{P}_{500Hz} = \frac{1}{2} A_0 \varrho c |\varphi|^2 \sqrt{1 - \left(\frac{f_l}{f_{500Hz}}\right)^2},$$  \hspace{1cm} (16.215)

the ratio of the two powers
\[ \frac{P_{500\text{Hz}}}{P_\infty} = \sqrt{1 - \left( \frac{f_l}{500\text{Hz}} \right)^2}. \] (16.216)

As required by the task, at this frequency:

\[ 10 \cdot \log_{10} \left[ \frac{P_{500\text{Hz}}}{P_\infty} \right] = 10 \cdot \log_{10} \left[ \sqrt{1 - \left( \frac{f_l}{500} \right)^2} \right] = -3 \text{ dB}. \] (16.217)

Substitution of \(2\pi \cdot f_l = \omega_l = \varepsilon c\) and \(c = 344 \text{ [m/s]}\) yields

\[ \sqrt{1 - \left( \frac{\omega_l}{2\pi \cdot 500} \right)^2} = \frac{1}{2}, \] (16.218)

\[ \frac{344 \varepsilon}{2\pi \cdot 500} = \sqrt{\frac{3}{4}}, \] (16.219)

so the flare coefficient \(\varepsilon\)

\[ \varepsilon = \frac{\sqrt{3} \cdot 500 \pi}{344} \text{ [m}^{-1}] = 7.91 \text{ [m}^{-1}]. \] (16.220)

**Problem 8.4**

**Focus**

Given a stepped tube composed of two sections with a pressure chamber.

**Task**

(a) Determine the function \(H(\omega) = \frac{q_m(\omega)}{q_g(\omega)}\)

(b) Find the poles of this function. Note that the termination impedance is assumed very small in comparison to the pipe resistance.

**Setup**

At the right end of the tube, \(A_2\), the end is open, so \(p_m = 0\). The transmission line equation in matrix form is given by:

\[ \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} \cos(\beta l) & j Z_L \sin \beta l \\ j \frac{1}{Z_L} \sin \beta l & \cos(\beta l) \end{bmatrix} \begin{bmatrix} p_2 \\ q_2 \end{bmatrix}. \] (16.221)
Solve

(a) Using the transmission line equation matrix:

\[
\begin{bmatrix}
    p_g \\
    q_g
\end{bmatrix} = [M_1][M_2] \begin{bmatrix}
    p_m \\
    q_m
\end{bmatrix},
\]

(16.222)

where \([M_1]\) is the transmission line matrix for the first tube, and \([M_2]\) is for the second tube.

This is the equation for the first tube:

\[
\begin{bmatrix}
    p_g \\
    q_g
\end{bmatrix} = \begin{bmatrix}
    \cos(\beta l_1) & j Z_{l_1} \sin(\beta l_1) \\
    j Z_{l_1} \sin(\beta l_1) & \cos(\beta l_1)
\end{bmatrix} \begin{bmatrix}
    p_0 \\
    q_0
\end{bmatrix},
\]

(16.223)

where the output of the first tube is \(p_0\) and \(q_0\). Therefore:

\[
q_g = p_0 \cdot j \frac{1}{Z_{l_1}} \sin(\beta l_1) + q_0 \cos(\beta l_1),
\]

(16.224)

Note that the termination impedance is assumed very small in comparison to the tube resistance, which means tube \(l_2\) is open with \(p_m \approx 0\). Therefore, the matrix for the second duct becomes

\[
\begin{bmatrix}
    p_0 \\
    q_0
\end{bmatrix} = \begin{bmatrix}
    \cos(\beta l_2) & j Z_{l_2} \sin(\beta l_2) \\
    j Z_{l_2} \sin(\beta l_2) & \cos(\beta l_2)
\end{bmatrix} \begin{bmatrix}
    0 \\
    q_m
\end{bmatrix}.
\]

(16.225)

Substitution of Eq.(16.225) into Eq.(16.223) yields

\[
\begin{bmatrix}
    p_g \\
    q_g
\end{bmatrix} = [M_1] \cdot [M_2] \begin{bmatrix}
    0 \\
    q_m
\end{bmatrix},
\]

(16.226)

with

\[
[M_1][M_2] = \begin{bmatrix}
    \cos(\beta l_1) & j Z_{l_1} \sin(\beta l_1) \\
    j Z_{l_1} \sin(\beta l_1) & \cos(\beta l_1)
\end{bmatrix} \begin{bmatrix}
    \cos(\beta l_2) & j Z_{l_2} \sin(\beta l_2) \\
    j Z_{l_2} \sin(\beta l_2) & \cos(\beta l_2)
\end{bmatrix},
\]

(16.227)

\[
\begin{bmatrix}
    p_g \\
    q_g
\end{bmatrix} = \begin{bmatrix}
    N_{11} \\
    N_{21} \cos(\beta l_1) \cos(\beta l_2) - Z_{l_2} Z_{l_1} \sin(\beta l_1) \sin(\beta l_2)
\end{bmatrix} \begin{bmatrix}
    0 \\
    q_m
\end{bmatrix},
\]

(16.228)

so that

\[
q_g = 0 \cdot N_{21} + \left[ \cos(\beta l_1) \cos(\beta l_2) - \frac{Z_{l_2}}{Z_{l_1}} \sin(\beta l_1) \sin(\beta l_2) \right] q_m,
\]

(16.229)

which leads to

\[
H(\omega) = \frac{q_m}{q_g} = \frac{1}{\cos(\beta l_1) \cos(\beta l_2) - \frac{Z_{l_2}}{Z_{l_1}} \sin(\beta l_1) \sin(\beta l_2)}.
\]

(16.230)

(b) The poles can be found when the denominator of above equation becomes 0,
\[ \cos(\beta l_1) \cos(\beta l_2) \left[ 1 - \frac{Z_{l_2}}{Z_{l_1}} \tan(\beta l_1) \tan(\beta l_2) \right] = 0, \quad (16.231) \]

which indicates: the first condition: \( \beta l_1 \) or \( \beta l_2 = \frac{\pi}{2} \).

the second condition: \( \frac{Z_{l_2}}{Z_{l_1}} \tan(\beta l_1) \tan(\beta l_2) = 1 \), or \( \frac{A_1}{A_2} \tan(\beta l_1) \tan(\beta l_2) = 1 \).

Upon assumption that \( l_1, l_2 << \lambda \), then \( \tan(\beta l_1) \approx \beta l_1 \), \( \tan(\beta l_2) \approx \beta l_2 \),

\[ \frac{A_2}{A_1} = \beta^2 l_1 l_2. \quad (16.232) \]

With \( \beta = \omega/c \), the second condition for poles will be

\[ \omega^2 = \frac{c^2}{l_1 l_2} \frac{A_2}{A_1}, \quad (16.233) \]

\[ \omega = c \sqrt{\frac{A_2}{l_1 l_2 A_1}}. \quad (16.234) \]

**Problem 8.5**

**Task**

Given a rigidly terminated tube with its diameter much smaller than the wave length of traveling wave, determine the field impedance at the surface indicated by dot-line inside the tube.

**Focus**

The tube diameter is much smaller than the wave length of the traveling wave from left toward the rigid termination, so the plane wave condition is fulfilled. From the surface indicated by the dot-line to the rigid termination, the tube segment of length \( D \) is under consideration. The termination is rigid, it follows that the volume velocity \( q_t \) or the particle velocity \( v_t \) at the termination must be zero.

**Setup**

Take Eq.(8.48) for this specific task

\[ \begin{bmatrix} P_s \\ q_s \end{bmatrix} = \begin{bmatrix} \cos \beta D & j Z_D \sin \beta D \\ j \frac{1}{Z_D} \sin \beta D & \cos \beta D \end{bmatrix} \begin{bmatrix} P_t \\ q_{t_r} \end{bmatrix}, \quad (16.235) \]

where the sound pressure and the volume velocity are noted by \( P_s \) and \( q_s \) at the surface indicated by the dot-line, also the tube is assumed to have a cross-section area \( A \), and the segment length \( D \) with a segment impedance \( Z_D = Z_w/A = \rho c/A \), and \( q_s = v_s A \).
Solve

Since $q_t = 0$ or $v_t = 0$ at the rigid termination, equation (16.235) above becomes

\[
\begin{bmatrix}
  p_s \\
  q_s
\end{bmatrix} =
\begin{bmatrix}
  \cos \beta D & j Z_D \sin \beta D \\
  j \frac{1}{Z_D} \sin \beta D & \cos \beta D
\end{bmatrix}
\begin{bmatrix}
  p_t \\
  0
\end{bmatrix},
\]

(16.236)

where the sound pressure $p_t$ at the termination is an unknown, yet maximum value. This leads to

\[
\begin{bmatrix}
  p_s \\
  q_s
\end{bmatrix} =
\begin{bmatrix}
  \cos(\beta D) p_t \\
  j \frac{1}{Z_D} \sin(\beta D) p_t
\end{bmatrix}.
\]

(16.237)

The surface (field) impedance $Z_s$ at the surface (indicated by dot-line) under consideration

\[
Z_s = \frac{p_s}{q_s} = \frac{p_s}{q_s/A} = \frac{A \cos(\beta D)}{j \sin(\beta D)/Z_D}.
\]

(16.238)

With $Z_D A = Z_w = \rho c$, and $\beta = \omega/c$, we finally obtain

\[
Z_s = \frac{p_s}{q_s} = -j Z_D A \frac{\cos(\beta D)}{\sin(\beta D)} = -j \rho c \cot \left( \frac{\omega D}{c} \right).
\]

(16.239)