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BAYESIAN ANALYSIS IN CHARACTERIZING SOUND ENERGY DECAY: A QUANTITATIVE THEORY OF INFERENCE IN ROOM ACOUSTICS

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ABSTRACT
Concert hall designs that incorporate coupled hard chambers have been drawing increasing attention in the room-acoustics community. One important task in analyzing the acoustics of coupled spaces is the evaluation of decay parameters associated with the single- or double-sloped decay characteristics of Schroeder decay functions derived from experimentally determined room impulse responses. Traditionally, however, characterization of double-sloped energy decay from room impulse responses has been very challenging. This work applies Bayesian probabilistic inference to cope with the demands of estimating diverse decay parameters from Schroeder decay functions. Recent effort in room-acoustic energy decay analysis using Bayesian probability theory demonstrates that Bayesian probability theory, along with the Ockham's Razor inherently embedded within Bayesian framework can help acousticians to quantify single-slope or multi-slope energy decay characteristics. This paper discusses decay order determination, estimation of decay parameters and their uncertainties within Bayesian framework.

INTRODUCTION
Reverberation times are the most common quantity to characterize sound energy decays in enclosures, assuming that the sound energy decays more or less exponentially. Yet sometimes, the sound energy can also decay non-exponentially as found already by Eyring [1], sound energy decay functions on a logarithmic scale are not generally linear for coupled spaces having different natural reverberation times or even for a single-space room with nonuniformly distributed absorption and no diffusing scheme. In effect, the sound energy decay in a single-space room can be of multi-rate character. Sound energy decays are commonly obtained via room impulse responses through acoustical measurements or computer-simulation techniques followed by Schroeder's backward integration [2]. Additional analysis procedures are needed to infer relevant parameters for the purpose of energy decay characterization. Recent works have established a parametric model [3-4] for the Schroeder decay functions according to the nature of Schroeder's backward integration.
When analyzing sound energy decays given the Schroeder decay functions, we should first ask the question "how many decay rates (slopes) are there in the data?" before any effort in estimating reverberation / decay times and related parameters. ISO 3822 [5] states "in cases where the (logarithmic) decay curve is not a straight line a unique reverberation time cannot be said to exist." If there would be a double-slope decay in the data, one would still ask the data to provide the parameters associated with a single-slope decay or with a triple-slope decay, one could not get right answer. More subtly, if differences between a double-slope and a triple-slope representation are small, what is a justifying reason for choosing the double-slope representation? Taking this room-acoustic application as an example, this paper discusses how Bayesian probability theory as a quantitative theory of inference, is applied in room-acoustics applications. Using the established Schroeder decay models [3-4], the room-acoustics energy decay analysis requires two levels of model-based inference [6]; one level of the inference is the task of model comparison and selection (MCS) among a set of competing models. Another level of inference assumes that a particular model is true and then determines the posterior probability function (PPDF) for the model parameters. These PPDF are used to estimate the model parameters, so-called model-based parameter estimation (PE). This paper will briefly report our recent effort in Bayesian PE [4,7,8] and our on-going effort in Bayesian MCS [9-10] for the room-acoustic energy decay analysis.

DECAY MODELS AND MODEL SELECTION

Schroeder decay models

Steady-state sound energy decays after sound sources deceased are determined by Schroeder integration [2] from room impulse responses experimentally measured or numerically simulated for enclosures under test. $\mathbf{D} = [d_1, d_2, \ldots, d_K]^{Tr}$ represents Schroeder decay function data, $(\cdot)^{Tr}$ stands for matrix transpose, K is the total number of data points in $\mathbf{D}$. Schroeder decay function models have been established based on the nature of Schroeder's integration [4,8]:

$$\mathbf{D} = \mathbf{G}_m \mathbf{A}_m + \mathbf{e}\,, \qquad \text{(Eq.1)}$$

which approximates the data $\mathbf{D}$ with an error vector **e**. $\mathbf{A}_m$ is a column vector of $(m+1)$ *amplitude* coefficients, termed *linear parameter vector*. $\mathbf{G}_m$ is a matrix of $K \times (m+1)$. m is the number of decay rates or *model order*. The *j*th column of $\mathbf{G}_m$ is given by its matrix element

$$g_{kj} = \begin{cases} t_k - t_K & for \quad j = 0 \\ \exp(-13.8 \cdot t_k / T_j) - \exp(-13.8 \cdot t_K / T_j) & for\, j = 1,2,\ldots,m \end{cases}, \qquad \text{(Eq.2)}$$

where T_j is *j*th decay time to be determined for $1 \le j \le m$, $T_0 = \infty$. The quantity t_K represents the upper limit of Schroeder's integration, $0 \le k < K-1$. Recent works [3,4,7,8] have experimentally proven the validity of this model. Figure 1 shows one example of Schroeder decay functions evaluated from scale-model coupled spaces and its model function $\mathbf{G}_2\mathbf{A}_2$ with properly estimated model parameters.

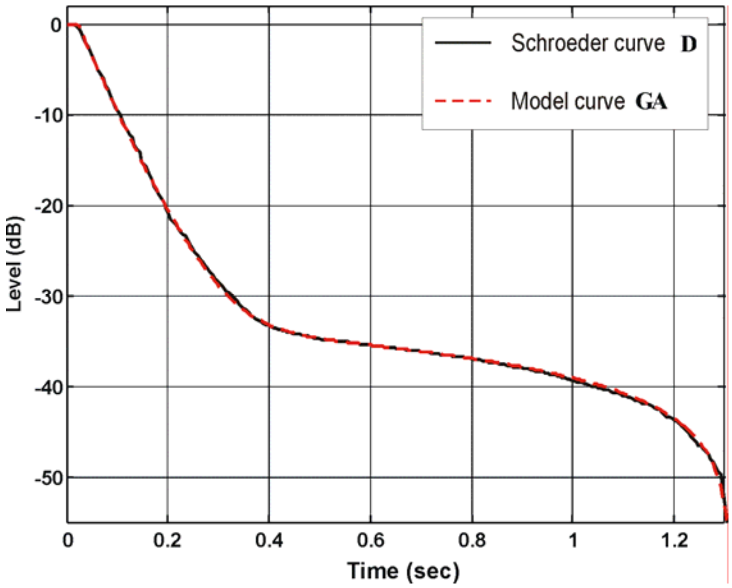

Figure 1.-Comparison between Schroeder decay function (**D**) measured in real halls and their model functions $\mathbf{GA}$. Sound energy decay (Schroeder) function measured in scale-model coupled spaces.

Model comparison and Ockham's razor

Once the Schroeder decay function data $\mathbf{D}$ are available, we consider m-order Schroeder decay model:

$$F_m = \mathbf{G}_m \mathbf{A}_m\,. \qquad \text{(Eq.3)}$$

The model comparison in the sound energy decay analysis is to determine which model the data are in favour among two competing models. However, the correct approach is NOT simply to choose the model that fits the data best: more complex models can always fit the data better

[6]. Bayes' theorem formulates the problem in terms of posterior probability density function

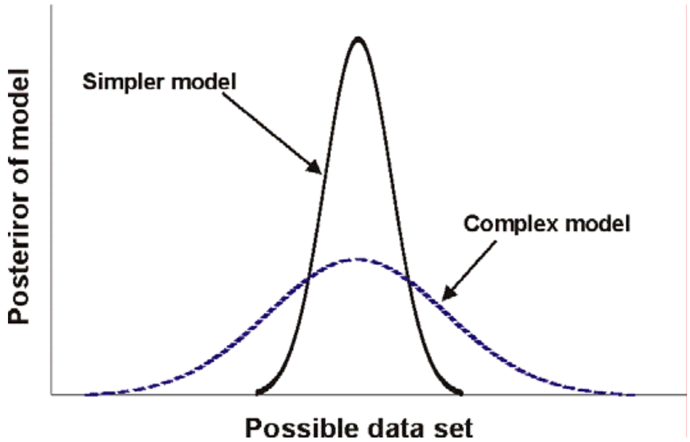

Figure 2.-Ockham's razor: among competing hypotheses, favour the simplest one. Bayesian inference automatically embodies Ockham's razor, which quantitatively penalizes over-parameterized models.

(PPDF) of model F_m:

$$p(F_m \mid \mathbf{D},\ I) = \frac{p(F_m \mid I)\, p(\mathbf{D} \mid F_m, I)}{p(\mathbf{D} \mid I)}\,. \quad \text{(Eq.4)}$$

Background information I includes the information that two (e.g. $m = 1{,}2$) models in eq.(2) through eq.(3) are competing to each other to represent the data reasonably well in a sense that the error vector is of finite value. These models can be compared on the basis of the following ratio [6]:

$$\frac{p(F_1 \mid \mathbf{D},\ I)}{p(F_2 \mid \mathbf{D},\ I)} = \frac{p(F_1 \mid I)}{p(F_2 \mid I)}\ \frac{p(\mathbf{D} \mid F_1, I)}{p(\mathbf{D} \mid F_2, I)}\,. \quad \text{(Eq.5)}$$

Table I.- Calculated model parameters including odds from a data set measured in Student Union (University of Mississippi), showing that the data prefer the 2nd model order [11].

Decay model order	1st order	2nd order	3rd order
Amplitude A_0 (dB)	-31.02	-33.39	-32.83
Amplitude A_1 (dB)	-5.44	-5.62	-6.65
Decay time T_1 (s)	0.93	0.87	0.86
Amplitude A_2 (dB)	--	-18.59	-12.51
Decay time T_2 (s)	--	2.47	0.92
Amplitude A_3 (dB)	--	--	-18.00
Decay time T_3 (s)	--	--	2.13
Log odds $\log_{10} O_{j2}$	-13.29	0	-0.48

To see how Bayesian model comparison incorporates Ochham's razor, consider the case where F_1 and F_2 can both fit the data equally well. The first (prior) ratio on the right-hand side in Eq. 5 indicates how much our initial beliefs favours model F_1 over F_2, the second (likelihood) ratio measures how well the available data were modelled by F_1 in comparison with F_2. Assigning equal prior $p(F_1 \mid I) = p(F_2 \mid I)$ would imply no subjective preference to either of models, leaving only the likelihood ratio on the right-hand side. If F_2 is higher order (more complex) model than F_1, it must spread its predictive probability $p(\mathbf{D}\,|\,F_2\,,\,I)$ more thinly over the data space than F_1 as shown in Fig.2. In the overlapped region of possible data set space where the data are compatible with both models, the simpler F_1 will become more probable than F_2. The ratio in Eq. 5 allows us to evaluate the plausibility of two alternative models in the light of data $\mathbf{D}$ [6]. The Bayesian formulation associated with Eq.5 inherently implements Ockham's razor: it favours simpler models with greater predictive power provided they give a good fit of the data. Bayesian model comparison automatically contains a logic mechanism for penalizing over-parameterized models [9]. However, if the data prefer a higher order model, its predictive power must have already overcome the Ockham's penalty. Our recent effort in applying quantitative methods of Bayesian MCS in room-acoustic analysis is documented in [10,11]. Table I lists results of odds values along with model parameters estimated from a room impulse response measured in Student Union at the University of Mississippi.

DECAY PARAMETER ESTIMATION

After selecting a decay model preferred by data, Bayesian theory formulates the PPDF of model parameters through the prior probability density and likelihood function via Bayes' theorem:

$$p(\mathbf{A},\mathbf{T}\,|\,\mathbf{D},I) = \frac{p(\mathbf{A},\mathbf{T}\,|\,I)\,p(\mathbf{D}\,|\,\mathbf{A},\mathbf{T},I)}{p(\mathbf{D}\,|\,I)}, \qquad \text{(Eq.6)}$$

where $p(\mathbf{D}\,|\,I)$ acts in the context of parameter estimation as a normalization constant. $\mathbf{T}$ is a vector matrix of *m* coefficients. $p(\mathbf{A},\mathbf{T}\,|\,I)$ is the prior distribution function of $\mathbf{A}$ and $\mathbf{T}$. Bayes' theorem in Eq. 6 represents how our prior knowledge $p(\mathbf{A},\mathbf{T}\,|\,I)$ is modified in the light

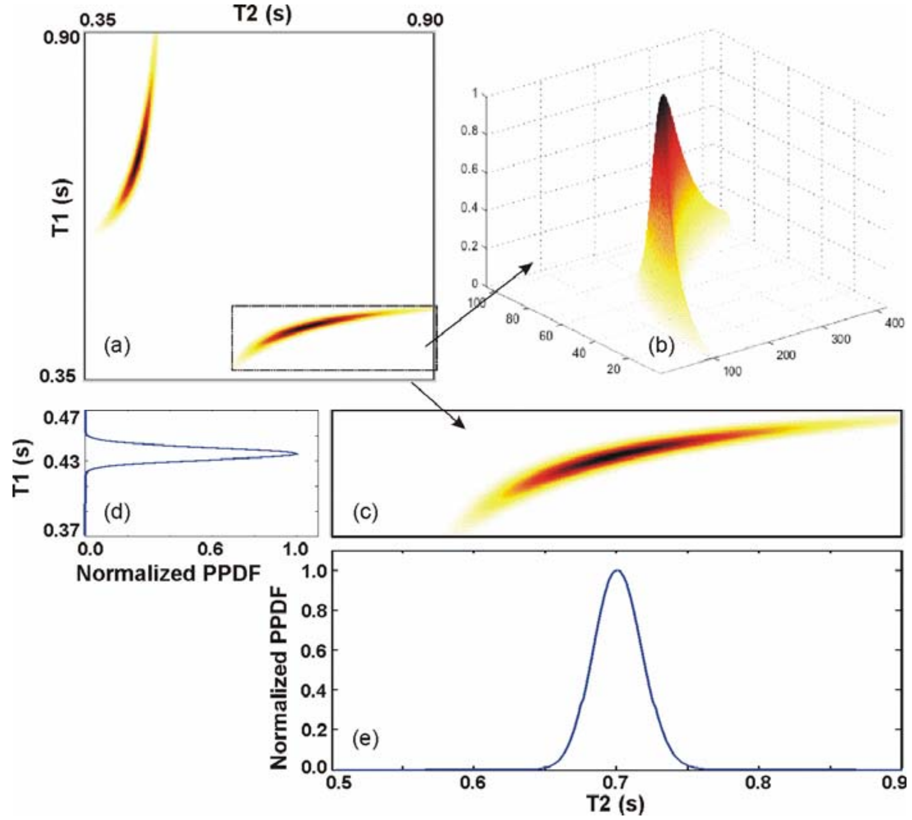

Figure 3.-Posteiror probability density function (PPDF) of decay times evaluated from a Schroeder decay function measured in scale-model coupled spaces. (a) PPDF in 2-D representation over $T_{1,2}$ within {0.35, 0.9} s. (b) PPDF in 3-D representation over a subspace evaluated with a grid of 108 x 420. (c) A-zoomed PPDF over T_1 within {0.37, 0.47} s, T_2 within {0.5, 0.9} s. (d), (e) projection of the line across the peak PPDF onto individual decay time axis.

of data through the likelihood function $p(\mathbf{D} \mid \mathbf{A}, \mathbf{T}, I)$. Background information I here includes that the Schroeder decay model in Eqs. 1-2 describes the data $\mathbf{D}$ reasonably well so that all errors in $\mathbf{e}$ are bounded by a finite value. Given finite errors and a reasonable model as only available information, application of the principle of the maximum entropy [4] assigns a Gaussian distribution to the likelihood function $p(\mathbf{D} \mid \mathbf{A}, \mathbf{T}, I)$ and an independence to errors e_i from each other, so that

$$p(\mathbf{D} \mid \mathbf{A}, \mathbf{T}, \sigma, I) = \left(\sqrt{2\pi}\sigma\right)^{-K} \exp\left(-\frac{\mathbf{e}^{Tr}\mathbf{e}}{2\sigma^2} \right), \qquad \text{(Eq.7)}$$

with a finite, but unspecified error variance σ^2. Probability theory provides us with rules of relating and manipulating the quantities, leading to an analytically tractable PPDF in form of the student-T distribution:

$$p(\mathbf{T} \mid \mathbf{D}, I) \propto \left(\mathbf{D}^{Tr}\mathbf{D} - \mathbf{q}^{Tr}\mathbf{q}\right)^{-(m-K)/2}, \qquad \text{(Eq.8)}$$

with $\mathbf{q} = \mathbf{Q}^{Tr}\mathbf{D}$ and $\mathbf{Q} = \mathbf{G}\mathbf{E}\mathbf{\Delta}^{-1}$ [8]. Figure 3 illustrates a PPDF (Eq. 8) over parameter space $\{T_1, T_2\}$ using Schroeder decay model order 2 evaluated from a room impulse response measured in scale-model coupled spaces. Localizing maximum values of the PPDFs, so-called maximum *a posterior* (MAP) approaches [4,8] or probabilistic approaches based on Markov chain Monte Carlo (MCMC) methods [7,10,11] have been used in room-acoustic energy decay analysis, where the uncertainties of inferred decay parameters can also be quantitatively estimated [7] through a covariance matrix [C_{ij}] with its expected matrix element:

$$\left\langle C_{ij} \right\rangle \approx \frac{\sum_{r=1}^{R} (T_{ir} - \langle T_i \rangle)(T_{jr} - \langle T_j \rangle) u(\mathbf{T}_r)}{\sum_{r=1}^{R} u(\mathbf{T}_r)}, \qquad \text{(Eq.9)}$$

with

$$u(\mathbf{T}_r) = \frac{p(\mathbf{T}_r \mid \mathbf{D}, I)}{g(\mathbf{T}_r)}, \qquad \text{(Eq.10)}$$

$g(\mathbf{T}_r)$ is the random process used to draw R samples from $p(\mathbf{T} \mid \mathbf{D}, I)$ in Eq. 8, and $\langle T_i \rangle$ is expected mean value of decay time T_i. In this way the standard derivations of individual estimated parameters, and dependences between parameters can also be estimated from the covariance matrix [C_{ij}] via Eqs.9-10 [7].

REPARAMETERIZED MODELS FOR AURALLY-SIGNIFICANT ACOUSTIC COUPLING

Acoustic designers have incorporated coupled hard chambers in concert halls [12] to achieve, in addition to variable acoustics, nonexponential multiple-sloped decay characteristics of sound energy decay which meet the following conditions:

$$A_1 > A_2 >, \ldots, \text{ and } T_1 < T_2 <, \ldots. \qquad \text{(Eq. 11)}$$

The motivation of achieving such decay characteristics is that two completing, yet desirable features of perceived clarity and reverberance are believed to be simultaneously satisfied [12]. The corresponding acoustic coupling in achieving such decay characteristics will be ***aurally-significant***. If the conditions expressed in Eq. 11 cannot be met, e.g. 'double-slope' sound energy decays will become single-slope nature. Our recent effort in applying Bayesian inferential methods also includes incorporating these conditions (Eq.11) as prior information [11,13] into a re-parameterized Schroeder decay model. The challenges in assigning priors of model parameters and their estimations have been addressed in Ref. [11,13]. Note that the results given in Table I incorporate the conditions given by Eq. 11.

EVALUATIONS OF EXPERIMENTAL DATA

Bayesian probabilistic inference can provide us with model parameters, quantifying the sound decay process reliably. Table II lists a set of evaluation results from a room impulse response, measured in the Student Union at the University of Mississippi, across the octave frequency bands from 125 Hz to 2 kHz. Bayesian model comparison reveals that the data in 2 kHz band is of single-slope nature, therefore, only one single decay (reverberation) time is listed. In other 4 frequency bands, the model comparison confirms that the data prefer double slope decays. In addition to two decay times, or the decay time ratio, we are also interested in the ***decay level difference*** $\Delta L = 20\log(A_1 / A_2)$ first used in Ref. [4], rather than their individual values of A_1, A_2, respectively. The decay level difference quantifies how low the second decaying process characterized by T_2 is relative to the first one of T_1. Table II also lists the standard derivations (Std) of T_i. Successful application of Bayesian inferential analysis in sound energy decay analysis unambiguously demonstrates that other attempts including linear fit of different portions of decay functions for evaluating nonexponential decays are questionable, since a straightforward choice of predefined small portions introduces arbitrariness. Those scientifically unsounded attempts, therefore, should not be practiced any more.

Table II.- Calculated decay parameters across the octave bands from a room impulse response measured in the Student Union at University of Mississippi.

Band (Hz)	T_1 (s)	Std_1 (s)	T_2 (s)	Std_2 (s)	*Decay Time Ratio*	ΔL (dB)
125	0.71	6.87E-3	2.82	4.67E-2	3.93	2.60
250	0.68	2.91E-3	2.38	2.09E-2	3.46	3.54
500	0.99	3.20E-3	2.74	5.43E-2	2.75	7.13
1000	0.88	1.82E-3	2.63	4.22E-2	2.99	9.47
2000	0.92	3.03E-3	-	-	-	-

SUMMARY

This work applies Bayesian probabilistic inference to cope with the demanding tasks of estimating diverse decay parameters from Schroeder decay functions. In this application, two levels of inference are inevitably involved, decay model comparison and model-parameter estimation. Bayesian probability theory, relying on extensive uses of Bayes' theorem, includes all the rules of relating and modifying probabilistic quantities, it automatically embodies Ockham's Razor to evaluate whether single-slope or double-slope energy decay characteristics are favoured by the data. Once the decay model is selected, Bayesian decay parameter estimation will provide devise decay parameters, including decay times, decay level difference, and their uncertainties. Evaluations of experimental results using Bayesian approaches developed in our recent effort have demonstrated their effectiveness in the room-acoustics application.

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