

A specialized fast cross-correlation for acoustical measurements using coded sequences^{a)}

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In acoustics applications, binary maximal-length sequences and related sequences are increasingly used for acoustics system identification tasks. A number of coded sequences, such as binary maximal-length related sequences and ternary sequences possess two-valued or pulselike autocorrelation functions. It is this correlation property that is exploited in most of acoustical applications. However, the length of some of these sequences is not directly suitable for FFT-based cross-correlation algorithms. This paper explores using standard FFTs to calculate the cross-correlation between two periodic finite-length sequences of equal length, where the lengths of the sequences are not a power of 2. We apply our specialized correlation algorithm to analyze data collected in a room-acoustic environment to simultaneously obtain impulse responses between multiple sources and multiple receivers. © 2006 Acoustical Society of America. [DOI: 10.1121/1.2141236]

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I. INTRODUCTION

Maximal-length sequences and related coded signals have been increasingly used in acoustical applications.^{1–7} Recent works^{6,7} have reported on the application of binary maximal-length sequences (MLS) and MLS-related sequences in acoustical measurements such as using simultaneous sound sources. The simultaneous measurement technique can meet some critical requirements encountered in outdoor sound propagation research to investigate sound propagation near the ground surface in outdoor environment.^{8,9} This paper shows how periodic cross-correlations, in general, and channel impulse responses of acoustical systems, in particular, may be obtained via judicious application of conventional FFT techniques even in the case where the length of coded sequences to be evaluated is not suitable for conventional FFT algorithms.¹⁰ The practical significance of this algorithm for acousticians is to increase efficiency of the correlation measurement technique, particularly for those applications where a large number of simultaneous sound or vibration sources and receivers are needed.

Many types of coded sequences are suitable for use as excitation signals for this purpose; examples include MLS (binary and higher prime), and Legendre sequences.^{1,11} Each of these possesses a two-valued periodic autocorrelation function (PACF). In addition, MLS-related sequences, such as reciprocal MLS pairs,⁶ Gold, and Kasami sequences^{7,12} have a similar correlation property; their PACFs contain a single peak at zero-lag and low-valued side lobes relative to

the peak, and the periodic cross-correlation functions (PCCF) with other sequences can arrive at low values of the same amplitude as those in the side lobes of the PACF. It is this correlation property that is exploited in determining channel impulse responses and in many other applications, such as simultaneous multiple acoustic source measurements for room-acoustic investigations,⁶ and acoustic tomography.^{7,8} Performing the cross-correlation via discrete Fourier transforms is computationally inefficient [order of L^2 , denoted in this paper as $O(L^2)$], but the length (L) of some of these sequences is not directly suitable for FFT-based cross-correlation algorithms [$O(L \log_2 L)$]. The length of the binary periodic MLS is $L=2^N-1$, only one point less than the suitable length for FFT. Applying binary MLS in underwater experiments, Birdsall and Mertzger³ have explored the efficacy of zero-padding one point to obtain a suitable length for FFTs. They explore the effect of locating this additional point at all possible points within the sequence, but, in the end, the approach yields unacceptable errors with a spectral compensation.

There exists a fast algorithm, called the *fast MLS transform* (FMT), dedicated to binary MLS.^{6,13} The acoustical measurement techniques using MLS based on this fast algorithm has been frequently reported in major acoustical journals.^{2,6,14–16} The efficiency of the FMT is based on the fact that binary MLS matrix is permutationally similar to the Hadamard matrix^{17,18} in such a way that a fast Hadamard transform can be adopted for a high efficiency of cross-correlation calculation. This paper refers to this algorithm as the *Hadamard transform-based fast MLS-transform* (HT-FMT). The HT-FMT on acoustics responses to the MLS excitation directly results in impulse responses of the acoustical

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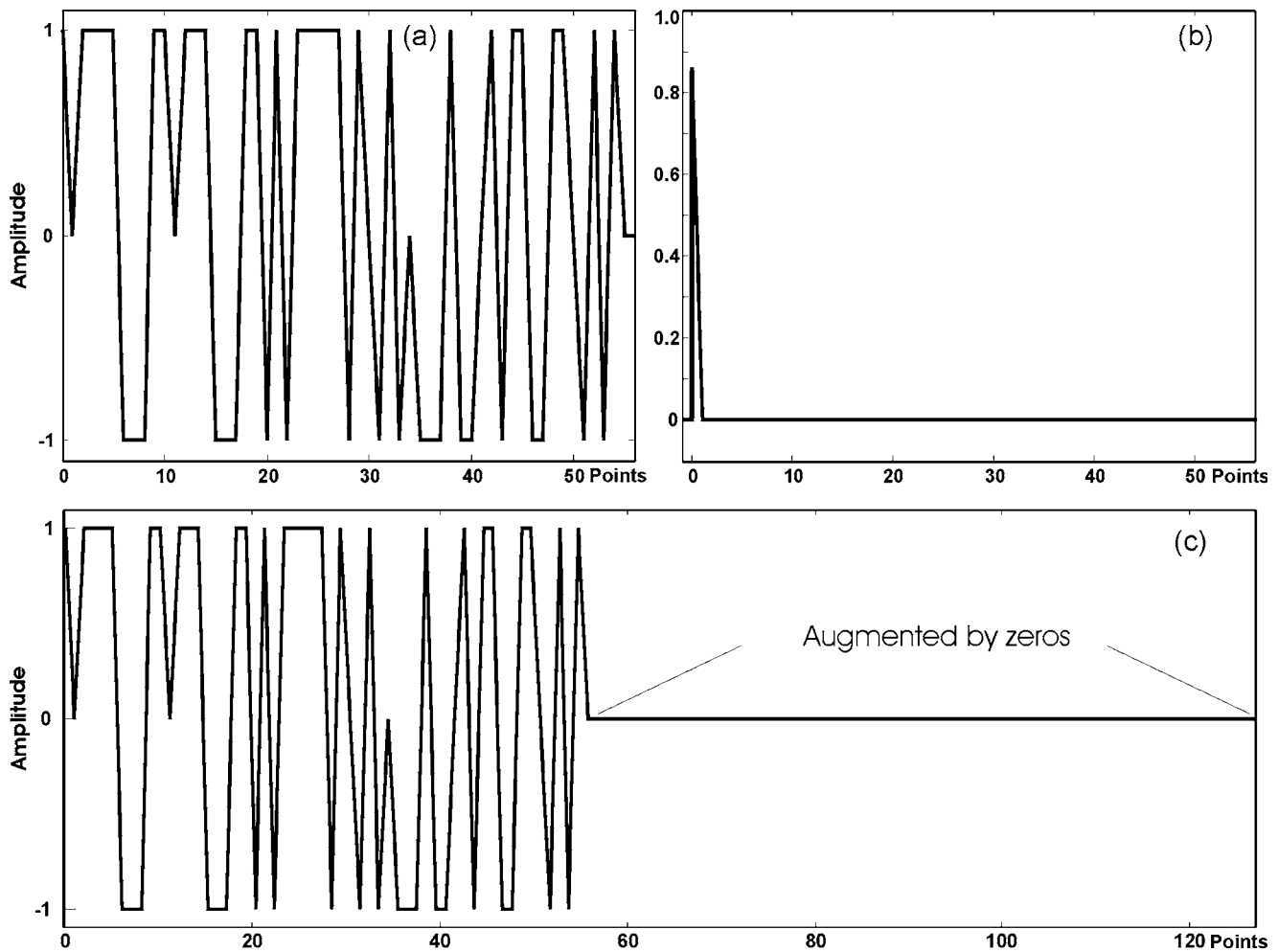


FIG. 1. (a) Ternary sequence of length 57. (b) Its autocorrelation function is two-valued (0.86, 0.0). (c) Augmented sequence of length 128 by zeros.

systems in the time domain. However, the HT-FMT can be applied only to binary M-sequences. Standard FFTs have not yet been successfully applied to cross-correlation of other sequences possessing the desired correlation property. While there exists techniques for efficiently computing FFTs of sequences of arbitrary length, such as Bluestein's algorithm (BFFT),¹⁹ we explore the use of standard FFTs in calculating the periodic cross-correlation between two periodic sequences of equal finite length, where the lengths of the sequences are not equal to a positive power of 2. In our approach we do not actually compute the FFTs of the original sequences, and, as we shall see, this yields a more efficient computational algorithm.

In the next section, we discuss the general problem of computing PCCFs and explain our approach to computing PCCFs of arbitrary-length (finite) sequences using FFTs. We then present a number of examples that illustrate the application of our approach. Finally, we draw some conclusions.

II. COMPUTATION OF CROSS CORRELATION FUNCTIONS

Let \mathcal{I} denote the set of integers and consider two sequences, $x[\cdot]=\{x[i], i \in \mathcal{I}\}$ and $y[\cdot]=\{y[i], i \in \mathcal{I}\}$ such that $x[i]=y[i]=0$ for $i \ni \{0, 1, \dots, L-1\}$ and $x[i]$ and $y[i]$ have arbitrary values for $i \in \{0, 1, \dots, L-1\}$. Define the sequences

$x_L[\cdot]$ and $y_L[\cdot]$, $I \geq L$ to be the sequences $x[\cdot]$ and $y[\cdot]$ repeated at intervals of length I . If $I > L$, then we refer to the sequences $x_L[\cdot]$ and $y_L[\cdot]$ as *augmented* sequences of $x_L[\cdot]$ and $y_L[\cdot]$, respectively. As an example, Fig. 1(a) illustrates a ternary sequence, say $t_L[\cdot]$ with $L=57$, and its augmented sequence $t_L[\cdot]$ with $I=128$ as shown in Fig. 1(c). In particular, define

$$x_L(i) = \sum_{j=-\infty}^{\infty} x[i-jL] \quad \text{and} \quad y_L(i) = \sum_{j=-\infty}^{\infty} y[i-jL]. \quad (1)$$

We wish to determine the periodic cross-correlation between $x_L[\cdot]$ and $y_L[\cdot]$, where L cannot be represented as a positive power of 2. In order to do this, we make use of the augmented sequences.

It is well-known²⁰ that the periodic cross-correlation

$$R_{x_L, y_L}[\cdot] = \mathcal{D}^{-1}(X_L[\cdot]Y_L^*[\cdot]), \quad (2)$$

where \mathcal{D} denotes the discrete Fourier transform (DFT) operator and $X_L[\cdot]=\mathcal{D}(x_L[\cdot])$ and $Y_L^*[\cdot]=\mathcal{D}^*(y_L[\cdot])$. Y^* stands for complex conjugate of Y . Similarly, if $I=2^N$ for some positive integer N , then

$$R_{x_l, y_l}[\cdot] = \mathcal{F}^{-1}(X_l[\cdot]Y_l^*[\cdot]), \quad (3)$$

where \mathcal{F} denotes the fast Fourier transform (FFT) operator, \mathcal{F}^{-1} the inverse FFT (IFFT), and $X_l[\cdot] = \mathcal{F}(x_l[\cdot])$ and $Y_l^*[\cdot] = \mathcal{F}^*(y_l[\cdot])$. The FFT is significantly more efficient computationally than the DFT $\{O[L \log_2(L)]$ versus $O(L^2)\}$. But if L does not have the form 2^N for some positive integer N , then the standard FFT cannot be used directly. We demonstrate below a technique for using the standard FFT as a primary computational tool for arbitrary values of L .

Let $R_{x,y}(k)$ denote the cross-correlation between $x[i]$ and $y[i]$ at offset k ; that is,

$$R_{x,y}[k] = \sum_{i=-\infty}^{\infty} x[i]y[i+k] = \sum_{i=\max\{0,-k\}}^{\min\{L-1,L-1-k\}} x[i]y[i+k], \quad (4)$$

where (4) follows from the fact that $x[i]$ and $y[i]$ are zero for $i \ni \{0, 1, \dots, L-1\}$. Now, if $k < 0$, then $\max\{0, -k\} = -k$ and $\min\{L-1, L-1-k\} = L-1$. Similarly, if $k \geq 0$, then $\max\{0, -k\} = 0$ and $\min\{L-1, L-1-k\} = L-1-k$, thus

$$R_{x,y}[k] = \begin{cases} \sum_{i=-k}^{L-1} x[i]y[i+k] & \text{for } k < 0, \\ \sum_{i=0}^{L-1-k} x[i]y[i+k] & \text{for } k \geq 0. \end{cases} \quad (5)$$

Thus, we readily see that $R_{x,y}[k] = 0$ for $k \ni \{0, \pm 1, \dots, \pm(L-1)\}$.

Now, define $R_{x,y_l}[k]$ to be the cross-correlation between $x[i]$ and $y_l[i]$ at offset k , it follows from (4) that

$$R_{x,y_l}[k] = \sum_{j=-\infty}^{\infty} R_{x,y}[k-jI]. \quad (6)$$

Similarly, since the cross-correlation of two sequences, $a[\cdot]$ and $b[\cdot]$, each having period I is defined as

$$R_{a,b}[k] = \lim_{\ell \rightarrow \infty} \frac{1}{2\ell I} \sum_{i=-\ell I}^{\ell I-1} a[i]b[i+k], \quad (7)$$

we find that

$$R_{x,y_l}[k] = \sum_{j=-\infty}^{\infty} R_{x,y}[k-jI]. \quad (8)$$

Now, let us determine the value of $R_{x,y_l}[k]$ over the period $k \in \{0, 1, \dots, I-1\}$ in terms of $R_{x,y}[k]$. Since $R_{x,y}[k] = 0$ for $k \ni \{0, \pm 1, \dots, \pm(L-1)\}$ and $I \geq L$, it follows that with $j < 0$, $R_{x,y}[k-jI] = 0$ for $k \geq 0$. Similarly, with $j > 1$, $R_{x,y}[k-jI] = 0$ for $k < I$. Therefore, we find that for $0 \leq k \leq I-1$,

$$R_{x,y_l}[k] = R_{x,y}[k] + R_{x,y}[k-I]. \quad (9)$$

In particular, if $I \geq 2L$, then (9) can be written as follows:

$$R_{x,y_l}[k] = \begin{cases} R_{x,y}[k], & 0 \leq k \leq L-1, \\ 0, & L \leq k \leq I-L, \\ R_{x,y}[k-I], & I-L < k \leq I-1. \end{cases} \quad (10)$$

Equation (10) can be rewritten as

$$R_{x,y}[k] = \begin{cases} R_{x,y_l}[I+k], & -(L-1) \leq k \leq -1, \\ R_{x,y_l}[k], & 0 \leq k \leq L-1, \\ 0 & \text{else.} \end{cases} \quad (11)$$

The point of (10) and (11) is that if I is chosen to be at least $2L$, then $R_{x,y_l}[\cdot]$ can be obtained from $R_{x,y}[\cdot]$. Thus, if we choose $I = 2^N$, where N is the minimum value yielding $I \geq 2L$, then $R_{x,y_l}[\cdot]$ can be computed from (3). Next, $R_{x,y}[\cdot]$ can be computed using (11), and finally, $R_{x,y_l}[\cdot]$ can be computed from (10) by using L in place of I .

As an example, we take a ternary sequence $\{t\} = \{1 \ 0 \ 1 \ 1 \ 1 \ 1 \ -1 \ -1 \ -1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ -1 \ -1 \ -1 \ 1 \ 1 \ -1 \ 1 \ -1 \ 1 \ 1 \ 1 \ 1 \ -1 \ 1 \ 0 \ -1 \ 1 \ -1 \ 0 \ -1 \ -1 \ -1 \ 1 \ -1 \ -1 \ 0 \ 1 \ -1 \ 1 \ 1 \ -1 \ -1 \ 1 \ 1 \ 0 \ -1 \ 1 \ -1 \ 1 \ 0 \ 0\}$ (Ref. 21) of length $L = 57$ as shown in Fig. 1(a) to demonstrate the described method in calculating its periodic autocorrelation function. The sequence $\{t\}$ contains 8 zeros in its period, so the zero-lag autocorrelation value amounts to $(57-8)/57 = 0.86$. Its periodic autocorrelation function is a two-valued function of $\{0.86, 0, 0\}$ as shown in Fig. 1(b). After augmenting the original sequence to $\{t_{128}\}$ with $t_i = 0$ for $i \ni \{0, 56\}$ as shown in Fig. 1(c) we can find from (3),

$$R_{t_{128}, t_{128}}[\cdot] = \mathcal{F}^{-1}(T_{128}[\cdot]T_{128}^*[\cdot]), \quad (12)$$

as shown in Fig. 2(a). Figure 2(a) clearly demonstrates that a cross-correlation between sequences of unsuitable length using a direct FFT-method in (3) and (12) will not provide correct results no matter how many zeros are padded to the sequences in order to get a suitable length for FFT operations. For this reason, we find from (11),

$$R_{t,t}[k] = \begin{cases} R_{t_{128}, t_{128}}[128+k], & -56 \leq k \leq -1, \\ R_{t_{128}, t_{128}}[k], & 0 \leq k \leq 56, \\ 0 & \text{else.} \end{cases} \quad (13)$$

Finally, we find from (11), for $0 \leq k \leq 56$

$$R_{t,t}[k] = R_{t_{128}, t_{128}}[k] + R_{t_{128}, t_{128}}[k+128-57]. \quad (14)$$

The two terms on the right-hand side of (14) are shown in Fig. 2(b), the addition of which results in exactly the correct PACF as shown in Fig. 1(b).

The described algorithm can be summarized in the following three major steps:

- (1) Augment the MLS-related sequences and other sequences $m[\cdot]$ of length L (not suitable for FFT) to $\hat{m}[\cdot]$ and the response $y[\cdot]$ to $\hat{y}[\cdot]$ by zeros, respectively to form a suitable length $I = 2^N$ for FFT operations, I is the first power of 2 that exceeds $2L$.
- (2) Cross-correlate the two augmented sequences of length I using the FFT,

$$\hat{h}[\cdot] = \mathcal{F}^{-1}\{\mathcal{F}(\hat{m}[\cdot])\mathcal{F}^*(\hat{y}[\cdot])\}. \quad (15)$$

- (3) Add the first L and the last L points in $\hat{h}[\cdot]$ to form the resulting $h[\cdot]$ of L points,

$$h_k = \hat{h}_k + \hat{h}_{k+I-L}, \quad 0 \leq k < L. \quad (16)$$

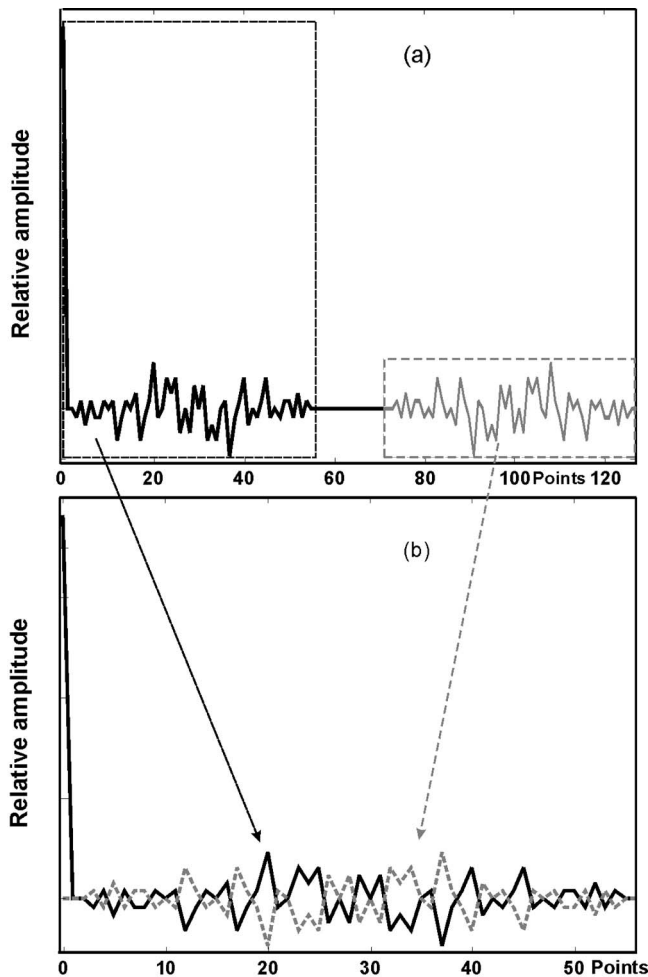


FIG. 2. Combining process illustrated using ternary sequence of length 57. (a) (Auto) correlation function evaluated using the augmented sequence to 128 points. (b) Combination (addition) of the first 57 and the last 57 points according to (14) results in the correct auto-correlation function of length 57 as shown in Fig. 1(b).

Using this method the FMT can be performed economically with a computational load of $O[L \log_2(L)]$. The FFT-based FMT algorithm is computationally more expensive than the HT-based FMT since the fast Hadamard transform consists solely of $O[L \log_2(L)]$ addition operations. However, the additional computation load will be less and less noticeable with the modern, fast developing computer technology. More importantly, it is of practical significance when other digital sequences have to be applied in acoustical experiments as the next section elaborates.

We emphasize here that our contribution is an algorithm to find the cross-correlation of two sequences of arbitrary length, not to find the FFTs of the original sequences. That is, our procedures do not yield FFTs of the original sequences, and consequently, we do not add anything whatever to the literature on computation of FFTs of sequences of arbitrary length. By contrast, the BFFT algorithm¹⁹ is specifically designed to compute the FFT of a sequence of arbitrary length. But, the BFFT algorithm requires three applications of the standard FFT algorithm to a sequence of the same length as our extended sequence in addition to miscellaneous convolutions to produce the coefficients. Thus, while the BFFT

algorithm is efficient for finding the FFTs of sequences such as our original sequences, its application to our particular case would result in a computational penalty of a factor of at least 3.

We note in passing that, depending on the value of L , it may be more efficient to compute the FFT of the original sequences by developing a specialized FFT routine. In particular, suppose that $N=r_1 r_2 \dots r_p$ where $\{r_1, r_2, \dots, r_p\}$ is the set of prime factors of N . Then, it is possible to develop a specialized FFT routine that requires $N \sum_{i=1}^p r_i$ operations,^{22,23} as opposed to N^2 for the DFT. For example, if $N=57$, then $N=3 \times 19$. Hence, the specialized FFT for a 57 point sequence requires $57(19+3)=1254$ operations, while using a DFT requires $57^2=3249$ and our technique based on standard FFTs requires $128 \times 14=1792$ operations. Similarly, if $N=255=3 \times 5 \times 17$, then a specialized FFT routine would require $255 \times (3+5+17)=6275$ operations while our technique based on standard FFTs requires $512 \times 18=9216$ operations. On the other hand, if $N=127$, then N is prime and a 127 point DFT routine requires $127^2=16129$ operations while our technique based on standard FFTs requires $256 \times 16=4096$ operations. Although there exist a dedicated fast Fourier transform for prime lengths,²⁴ which is suitable for binary Legendre sequences,¹¹ but only a few MLS-related sequences have a prime length, such as $2^{17}-1$, a so-called Mersenne number. Therefore, the prime-length FFT has limited applications. Thus, the value of developing a specialized FFT routine depends on the length of the sequence.

III. EXPERIMENTAL EXAMPLES

This section presents an experimental example to illustrate the application of the above-described algorithm in acoustical measurements. Unlike the application in spread spectrum technology where some desired code is used to modulate a digital data sequence before modulating a carrier, the coded signals here in acoustical applications are directly used to drive acoustic sources at a rate that is sufficiently high to cover the frequency range of interest according to the sampling theorem. In order to measure room impulse responses from two separate sound sources simultaneously in the Russell Sage Hall at Rensselaer Polytechnic Institute, Troy, New York, a pair of Kasami sequences of degree 20 were used to drive the sound sources simultaneously at a sampling frequency of 50 kHz. Two microphones were used to receive the steady-state room acoustic responses to the excitations. Figure 3 shows the first 1.2 s of the four room impulse responses of the channels between the two sound sources and the two microphones. Note that a preferred pair of MLSs or MLSs drawn from the maximum connect set⁷ might also be applied to this case, but with increasing number of simultaneous sources, these MLSs cannot meet the need, while enough number of Kasami or Gold sequences can be easily constructed to cope with the required number of sources.⁷ In addition, Kasami or Gold sequences are combined sequences, to which the HT-FMT can no longer be applied.

When calculating the cross-correlation between the acoustic (microphone) steady-state response to the Kasami-

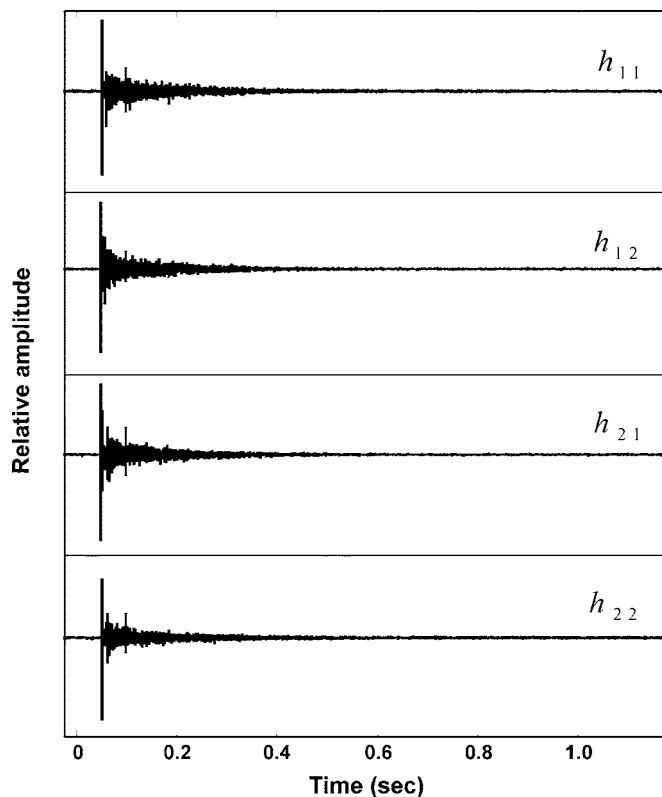


FIG. 3. Room impulse responses measured using the specialized FFT techniques in the Russell Sage Hall, RPI. Two separate sound sources are excited simultaneously by two Kasami sequences of degree 20. Two microphones measure the room acoustic responses to the Kasami sequence excitations. The specialized FFT-based cross-correlation algorithm results in four room impulse responses.

sequence and the original Kasami-sequence, the described algorithm is applied resulting in a desired room impulse response. This example exactly demonstrates the application of our algorithm to a case where the Hadamard-transform-based fast M-sequence transform cannot be applied and where use of the DFT is computationally grossly inefficient. It is easy to verify that the impulse response are identical to those obtained by direct cross-correlation.

IV. SUMMARY AND CONCLUSIONS

We have shown how periodic cross-correlations, in general, and channel impulse responses of acoustical systems, in particular, may be obtained via specialized application of conventional FFT techniques even in the case where the length of sequences to be evaluated are not of length p^N , where p is a prime. Conventional FFTs cannot directly process periodic cross-correlations of coded signals, such as ternary sequences, binary or higher prime maximum-length sequences, Gold, and Kasami sequences, which are increasingly used in acoustic system identification tasks. We have argued that our techniques can provide substantial computational improvements over direct DFT techniques in a practical situation, specifically when it is desirable or even mandatory simultaneously to obtain impulse responses between multiple sound/vibration sources and one or several receivers.

The application of Kasami sequences or Gold sequences⁷ is of particular interest when a large number of sound sources need to be driven simultaneously so that the measurement can be accomplished in the same time interval as those using a single source channel. As stated previously, Kasami and Gold sequences are combinations between binary maximal-length sequences and their derived sequences.⁷ The combination can easily provide a large number of sequences of the same length with desired correlation properties as suitable for simultaneous source excitations.

The ability to efficiently analyze room impulse responses using Kasami sequences is especially important in cases where it is desired to measure simultaneously the impulse responses at a point from a number of different sources. In that case, one can easily construct enough distinct Kasami sequences to meet the need,⁷ but neither the Hadamard-transform-based fast M-sequence transform nor the conventional FFT algorithm applied directly to the sequences can accomplish the processing. The specialized correlation algorithm described in this paper is simple to use. It involves only zero-padding of the original sequences, performing two conventional FFTs and one conventional IFFT, and simple addition to obtain the final result. This algorithm is also applicable to binary MLSs, so the fast MLS-transform can also be performed based on the specialized FFT-based correlation algorithm, then termed FFT-based FMT. The computation load of each single impulse response is, however, $3O(I \log_2 I)$. One can always carry out one FFT of the augmented excitation sequence once and store the results in memory before a large number of measurements using the same excitation sequence need to be performed so that the computation load is reduced to $2O(I \log_2 I)$.

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