Investigation of acoustically coupled enclosures using a diffusion-equation model

Ning Xiang, b) Yun Jing, and Alexander C. Bockman
Graduate Program in Architectural Acoustics, School of Architecture, Rensselaer Polytechnic Institute, Troy, New York 12180

(Received 5 June 2008; revised 11 June 2009; accepted 12 June 2009)

Recent application of coupled-room systems in performing arts spaces has prompted active research on sound fields in these complex geometries. This paper applies a diffusion-equation model to the study of acoustics in coupled-rooms. Acoustical measurements are conducted on a scale-model of two coupled-rooms. Using the diffusion model and the experimental results the current work conducts in-depth investigations on sound pressure level distributions, providing further evidence supporting the valid application of the diffusion-equation model. Analysis of the results within the Bayesian framework allows for quantification of the double-slope characteristics of sound-energy decays obtained from the diffusion-equation numerical modeling and the experimental measurements. In particular, Bayesian decay analysis confirms sound-energy flux modeling predictions that time-dependent sound-energy flows in coupled-room systems experience feedback in the form of energy flow-direction change across the aperture connecting the two rooms in cases where the dependent room is more reverberant than the source room.

© 2009 Acoustical Society of America. [DOI: 10.1121/1.3168507]

PACS number(s): 43.55.Br, 43.55.Ka [LMW]

Pages: 1187–1198

I. INTRODUCTION

A number of recently built performing arts venues contain one or more reverberant auxiliary rooms that are connected to the primary room (audience chamber) in such a way that there is an exchange of acoustic energy through an opening connection (coupling aperture). Such configurations constitute systems of coupled-rooms. In addition to enabling variable acoustics, one motivation behind use of coupled-rooms is the creation of particular kinds of non-exponential sound-energy decays,1–3 so-called double-slope decays.4 When sound-energy decays consist of a double-slope process with a rapid initial portion followed by a slower late portion, they are believed to simultaneously realize the desirable, yet competing perceptual attributes of clarity and reverberance. The subject of this paper is to apply a modeling technique based on diffusion equations to study acoustics in coupled-rooms.

Geometrical-acoustics (Refs. 5 and 6) and wave-acoustics approaches7,8 have been recently used for numerically or computationally modeling the behavior of sound fields in coupled-rooms. The wave-acoustics methods are considered the more rigorous of the approaches as they more fully represent the phenomenology of sound fields. However, they only yield analytic solutions for a few simple cases and are problematic in high-frequency regimes due to the high computational load. By contrast, geometrical-acoustics, in which sound fields are modeled by ensembles of non-interacting classical phonons, is an inherently high-frequency approach. Though it does not necessarily allow for a greater number of enclosure geometries to be solved numerically, it is well suited to various computational implementations. Furthermore, geometrical-acoustics methods provide a rigorous base from which to develop statistical models of sound fields. In recent work,6 the cone-tracing algorithm with randomized tail corrections used by the commercial software CATTACOUSTIC was modified in order to model coupled-rooms.

Due to its computational efficiency in comparison with existing geometrical-acoustics and wave-theoretical approaches, transport/diffusion theory has been actively applied in urban noise propagation and room-acoustics predictions.9–18 The calculation load of diffusion-equation modeling on currently available personal computers is on the order of seconds for steady-state simulations and 1 min or less for transient simulations for all numerical examples discussed in this paper. In a recent paper, Billon et al.12 applied diffusion equations in acoustically coupled-rooms. In their work, Billon et al.12 compared experimentally measured results in two coupled empty rooms with predicted results using the diffusion-equation model. The boundary conditions used for the diffusion equation in their work have certain limitations. As reported by Valeau et al.,11 when the absorption assigned is as high as 0.2, discrepancies were found. Furthermore, the double-slope decays modeled by their diffusion model are not properly quantified.

This paper advances the efforts by Billon et al.12 in several significant ways. First, this paper illustrates predicted sound-energy flows in two coupled-rooms. The correlation between the energy flow decay function and the steady-state energy decay function is discussed. Second, this work em-
employs an eighth scale-down system of two coupled-rooms to investigate the relevant aspects. Finally, the double-slope decays from the numerically modeled results and from the acoustical scale-modeling results are quantified within a Bayesian framework. The Bayesian probabilistic framework has been shown to estimate not only the decay parameters from a Schroeder decay model, but also to quantify uncertainties of decay time estimates and the interrelationship between multiple decay times.19

This paper is structured as follows: Section II briefly presents the governing equations of the diffusion-equation model including boundary conditions. Section III focuses on the sound pressure level (SPL) distributions and sound-energy decays in the coupled spaces. The experimental results obtained from the scale-model coupled-rooms are compared with the simulation results. Section IV elaborates on the energy flow in coupled spaces, including both the time-dependent energy flow directions and energy flow decays. Section V concludes the paper.

II. DIFFUSION-EQUATION MODELS

The diffusion-equation model is based on the assumption that the room(s) under study contains scattering objects that uniformly scatter the sound and have the same mean free path length of the room when walls are considered diffusely reflecting. Under this circumstance, walls are replaced with the scattering objects. Following the physical analogy with the diffusion of particles in a scattering medium, and assuming sound particles travel along straight lines at sound speed \( c \) and follow a certain statistical process (usually happens after early reflections) in the room(s) under investigation with diffusely reflecting walls,10 the sound-energy density \( w(\mathbf{r},t) \) as a function of location \( \mathbf{r} \) and time \( t \) is then governed by the diffusion equation, which describes the energy flow from a high-density area to low-density area

\[
\frac{\partial w(\mathbf{r},t)}{\partial t} - D \nabla^2 w(\mathbf{r},t) + cmw(\mathbf{r},t) = q(\mathbf{r},t), \quad \in V, \tag{1}
\]

where \( D = \lambda c / 3 \) is termed the diffusion coefficient with \( \lambda \) being the mean free path. The subroom denoted by domain \( V \) has a source term \( q(\mathbf{r},t) \), which is zero for any subdomain where no source is present. The term \( cmw(\mathbf{r},t) \) accounts for air dissipation in the room(s) with \( m \) being the absorption coefficient of air,18 and can be extended to account for absorption due to scattering objects inside the room(s).13 Equation (1) is the interior equation in domain \( V \), being subject to the boundary condition on the interior surface \( S \),

\[
D \frac{\partial w(\mathbf{r},t)}{\partial n} + cAw(\mathbf{r},t) = 0, \tag{2}
\]

where \( n \) is the out-going normal of the wall surface. The absorption term \( A \) can take the following forms:

\[
A_S = \frac{\alpha}{4}, \tag{3a}
\]

or

\[
A_M = \frac{\alpha}{2(2 - \alpha)}, \tag{3c}
\]

The term \( A_S \) has been used in room-acoustics predictions since Ollendorff in 1969.9 More recently, Jing and Xiang14 and Billon et al.15 independently proposed the absorption term \( A_E \) in Eq. (3b) for modeling cases where some surfaces of the room under test have a high absorption. Diffusion equations using the absorption terms \( A_S \) or \( A_E \) in Eqs. (3a) and (3b) are designated as the diffusion-Sabine model or diffusion-Eyring model, respectively.15 The Sabine-diffusion model11,16 is only suitable for room surfaces with absorption coefficients not larger than 0.2. The Eyring-diffusion model fails to predict acoustic behavior in a room whenever a portion of room surfaces features a high absorption coefficient of 1.0; in this case, the Eyring-diffusion model suffers from a singularity problem. Most recently, Jing and Xiang16 proposed the term \( A_M \) in Eq. (3c); they demonstrated that the diffusion equation with this modified absorption term in the boundary condition is theoretically grounded and can model high absorption for a small portion of surfaces. In addition, the diffusion-equation model inherently assumes that overall absorption in rooms under test must not be high.11,16

III. SOUND PRESSURE DISTRIBUTIONS AND ENERGY DECAYS

Preliminary experimental verification of the diffusion-equation model for coupled spaces has been made in Ref. 12. This section further compares the diffusion-equation model with experimental results, conducted in a scale-model of two coupled-rooms, for both SPLs and energy decay characteristics.

A. Experimental model

The coupled-rooms are implemented in a 1:8 scale-model made of 2 in. thick plywood as shown in Fig. 1(a). The interior surfaces have been covered with rocks, adhered
to the wood surfaces with glue, to promote diffuse reflections. Figure 1(b) illustrates the dimensions of the rooms in Cartesian coordinates, given in full-scale. The room where a miniature dodecahedron loudspeaker system is placed to excite sound fields is referred to as the main (primary) room, while the other room, connected by a transparent area, is referred to as the secondary room. The transparent area connecting the two rooms is referred to as the (coupling) aperture, and its size is \(3 \times 3\) \(m^2\) throughout the experiments. In the scale-model experiments, the omni-directional sound source at location \((-5, 2.4, 1.3)\) \(m\) radiates a continuous maximal-length sequence signal of length \(2^{19}-1\) points in one period at a sampling frequency of \(100\) \(kHz\), averaged over ten repetitions, while a 1/4 in. microphone is used as a receiver. Room impulse responses are measured throughout all experiments. The miniature dodecahedron loudspeaker system can cover a frequency range up to \(32\) \(kHz\). With a scale factor 1:8, the measured room impulse responses contain useful information up to \(4\) \(kHz\) in full-scale. The glued rocks on interior surfaces can be considered diffusely reflecting above \(1\) \(kHz\) in full-scale.

When decoupled, reverberation times of each room, termed natural reverberation times and denoted as \(T_1^p\) and \(T_2^p\), to distinguish them from the decay times \(T_1\) and \(T_2\) of the double-slope decays in coupled-rooms, are determined first separately at five different locations as shown in Fig. 2 as spatially averaged values. The absorption coefficients of the wall materials are estimated from the averaged reverberation time by inverting the Eyring equation. No drying air or nitrogen \(^{20}\) is used to eliminate the air dissipation in the volume, since the purpose of this measurement is to verify the diffusion-equation model, not to model any real space. The obtained absorption coefficients of the wall materials are expected to include the air absorption. Therefore, the air dissipation term in the diffusion-equation model is ignored. The overall averaged absorption coefficients are listed in Table I.

### B. SPL distributions

The recent work of Billon et al.\(^{12}\) has reported reasonable agreement when comparing the SPL distribution along one straight line perpendicularly across the aperture. The SPL results reported in Ref. 12 show better agreement for higher-frequency bands (1 and 4 \(kHz\)), than for the lowest one (250 \(Hz\)). This work compares the SPL distributions in the two coupled-rooms and across the aperture between the scale-modeling experiments and the diffusion-equation modeling in aspects beyond the previous work. The diffusion model is implemented within two quasi-cubic rooms coupled through the coupling aperture as shown in Fig. 1(b). Equations (1), (2), and (3c) were solved by a finite-element method using a total of 8000 linear Lagrange-type mesh elements. The mean free paths of the primary and secondary rooms amount to \(3.6\) and \(4.6\) \(m\), and the mean free times \(\lambda/c\) are \(10.5\) and \(13.4\) ms, respectively. The mean free time is the time determined by one mean free path, \(\lambda\).

The steady-state SPL distribution is determined after solving for \(w(r)\) from Eqs. (1), (2), and (3c). Figure 3 illustrates the sound source location, meshing elements, and the aperture along with the SPL distributions using a half-transparent pseudo-color scale.

For a better understanding of the SPL distributions near the aperture, Fig. 4 illustrates two sets of acoustical measurement results at 1.5–4 \(kHz\) (broad-band) in the scale-model coupled-rooms in comparison with the predicted SPL values by the diffusion model. The direct sound is excluded from the experimental data, leaving only the reverberant sound. The comparison along two straight lines (0.32 \(m\) \(\leq y \leq 4\) \(m\)) parallel to the aperture shows agreement to a degree similar to that reported in the work of Billon et al.\(^{12}\). To be precise, the vertical axis in Fig. 4 is scaled to detail variations within a 10-\(dB\) range. Most experimental results show deviations smaller than 1-\(dB\), with maximal variation of 2.4 \(dB\) occurring at a position in the secondary room behind the wall. The reasonable agreement between the modeled and

**TABLE I. Absorption coefficient of the materials used in the scale-model at different octave bands (in full-scale).**

<table>
<thead>
<tr>
<th>Frequency (kHz)</th>
<th>500 Hz</th>
<th>1 kHz</th>
<th>2 kHz</th>
<th>4 kHz</th>
<th>1.5–4 kHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary room</td>
<td>0.21</td>
<td>0.28</td>
<td>0.30</td>
<td>0.32</td>
<td>0.30</td>
</tr>
<tr>
<td>Secondary room</td>
<td>0.14</td>
<td>0.16</td>
<td>0.17</td>
<td>0.19</td>
<td>0.17</td>
</tr>
</tbody>
</table>

**FIG. 2.** Spatially averaged values of natural reverberation times measured at five different locations in the eighth scale-model rooms shown in Fig. 1(a), when the two rooms are acoustically separated (single-rooms). The bars indicate range of data.

**FIG. 3.** (Color online) SPL distributions by the diffusion model in the scale-model coupled-rooms as shown in Fig. 1: (a) Three-dimensional presentation with a half-transparent pseudo-colored scale showing finite-element meshing. (b) Two-dimensional presentation in the \(X-Y\) plane.
measured SPLs indicates that the modified diffusion model implemented within this work can be used for detailed discussion on the SPL distributions and other aspects of sound fields.

One important aspect is the transition of the SPLs caused by the receiver location changing from a region close to the opening of the coupling aperture to the solid wall, and the transition of the SPLs when the receiver goes across the aperture from the primary room into the secondary room. In Fig. 3 the SPL distributions in pseudo-color scale across the aperture need further elaboration. Figure 5 illustrates SPL distributions by the modified diffusion model. Six different curves are shown as a function of $y$ across the entire width of the rooms at $x = \pm 0.01, \pm 0.3, \pm 0.5, \pm 1.0, \pm 1.5, \pm 2.0$ m, and $z = 3$ m [see Fig. 1(b) for definition of the coordinate]. As for SPLs at $x = \pm 0.01$ m, being close to the wall featuring the aperture, the SPLs near the edges of aperture from the aperture opening to the solid wall undergo a sudden change while the SPLs from the primary room to the secondary room within a region close to the aperture opening exhibit continuities. Similar results are also found by solving the diffusion-equation model when the secondary room is equally or more absorbent than the primary room. In the primary room, the SPLs toward the solid wall beyond the edge of the aperture increase abruptly, while in the secondary room they decrease abruptly.

### C. Sound-energy decays

For time-dependent solutions of the diffusion equation, the source term $q(r, t)$ on the right-hand side of Eq. (3) can be assigned as

\begin{equation}
q(r, t) = E_0 \delta(t),
\end{equation}

where $\delta(t)$ is the Dirac impulse, assuming an omnidirectional sound source with a sound-energy density $E_0$ at $t=0$ throughout a tiny, but finite volume $V_s$ occupied by the source located at $r_0\rightarrow r_s$. The solution of Eqs. (1), (2), and (3c) at any location $r$ in subdomain $V$ excluding $V_s$ represents an energy (density) room impulse response $w(r, t)$, exclusive of the direct sound. Its energy-time function (ETF) [usually known as energy-time curves (ETCs)] can be expressed as

\begin{equation}
L_p(r, t) = 10 \log_{10} \left( \frac{w(r, t) p_0^2}{P_{ref}^2} \right),
\end{equation}

where $\rho$ is the air density, $c$ is speed of sound, and $P_{ref}$ equals $2 \times 10^{-5}$ Pa. For sound-energy decay analysis, however, Schroeder-integration \cite{Schroeder} of the energy room impulse response yields an approximation of the steady-state sound-energy decay.
FIG. 7. Two energy-time curves and the corresponding, normalized Schroeder decay curves calculated using Eq. (6), predicted from the diffusion model. A sketch is included to show the receiver positions \((R_1, R_2)\), respectively, in the modeled coupled-rooms.

Distinctly different decay constants, the Schroeder-integration will yield decay terms with different decay parameters as defined in the following (Sec. III D).

Within proportionate spaces with single-slope decay, slopes of ETF curves (ETC) are similar to those of Schroeder decay functions, but when received at positions far from the sound source in elongated rooms, flat rooms, or at positions in the secondary coupled-room, where the direct sound cannot directly reach the sound receiver, the ETF curves will show a profound convex curvature at the initial part (see the ETF curve at \(R_2\) in Fig. 7). A direct usage of the ETF curves for reverberation time evaluation, particularly for early decay time estimation, will lead to biased results. More importantly, for energy decay analysis in coupled-rooms, where double-slope decay characteristics are often expected, the difference will be profound as shown in Fig. 7. At receiver \(R_1\) the sound energy exhibits double-slope decay characteristics. A comparison between the energy-time curve at \(R_1\) and its corresponding Schroeder decay curve shows a clear difference at the initial part, resulting in different decay parameters.

Another option for obtaining the desired sound-energy decay from the diffusion model is to assign a switch-off signal to the source term \(q(r_s, t);\) however, this is much more time-consuming.

D. Quantifying double-slope characteristics

To quantify double-slope characteristics of Schroeder decay functions either experimentally measured or diffusion-model predicted, Xiang and Goggans first proposed two decay times and a decay-level difference \(\Delta L\). Using these parameters as calculated by the software developed by Xiang and Goggans, Bradley and Wang carried out subjective tests using the data generated by a geometrical-acoustics based modeling software. More recently, Meissner also used similar parameters to quantify his modeled energy decays based on wave-equation based method. Figure 8(a) illustrates...
Bayesian analysis\cite{4,19,26} yields decay parameters associated with the three model terms in this case
\begin{equation}
F(A, T, t) = A_0(t_{\text{max}} - t) + A_1 \exp\left(-\frac{13.8t}{T_1}\right) \\
+ A_2 \exp\left(-\frac{13.8t}{T_2}\right),
\end{equation}
as plotted in Fig. 8(a) with $t_{\text{max}}$ being the time limit corresponding to the upper limit of Schroeder integration. Figure 8(b) illustrates a magnified view of the first 20 dB. As recommended by ISO 3382,\textsuperscript{27} the first 5 dB should be excluded for analysis purposes, particularly using the experimentally measured data; because the early portion of the energy decay represents only a small number of short-delay paths, it cannot be modeled by exponential decays. The data analysis from experimental measurements illustrated in Fig. 8 (as well as all other data analyses throughout this work) is undertaken from −5 dB to the end of the data record. Bayesian analysis is able to provide the model parameters in Eq. (8); namely, the amplitude parameters $(A_1, A_2)$, decay times $(T_1, T_2)$, associated uncertainties, and mutual dependence. $A_0$ is associated with the background noise in the room impulse response, actually a nuisance parameter, being a necessary part of the model, but irrelevant for the energy decay analysis. In addition to the individual decay times, architectural acousticians are interested in the relative amplitude $A_2$ with respect to $A_1$ for double-slope cases, rather than their individual values.\textsuperscript{7} The decay-level difference $\Delta L_S$ in decibels, first used by Xiang and Goggans,\textsuperscript{4} is defined as
\begin{equation}
\Delta L_S = 20 \log_{10}(A_2/A_1) \text{[dB] (9)}.
\end{equation}

Figure 8(b) illustrates this definition and further indicates the reason that the reverberation analysis is usually undertaken using the data segment from $S=-5$ dB point along the Schroeder decay function, particularly from the experimentally measured data or from geometrical-acoustics modeling as recommended by ISO 3822. The decay-level difference $\Delta L_S$ quantifies how low the second decaying process characterized by $T_2$ is relative to the first one of $T_1$. The value $S$, marking the starting point along the decay function, is subject to choice. Other than $S=-5$ dB, purposely taken to exclude a small, but bumpy portion owing to discrete nature predominately associated with early reflections, $S$ can take on a value to exclude an even smaller portion at the beginning of the decay function, such as $S=-0.5$ dB for those modeled by the diffusion-equation model as discussed in Sec. III C. These decay functions exhibit a smooth curve after one “mean free time.”

In Fig. 8(b) two straight lines corresponding to two decay slopes in logarithmic presentation can be determined as follows:
\begin{equation}
y_j = a_j + b_j t,
\end{equation}
with $a_j = 10 \log_{10}(A_j)$, $b_j = -10(13.8/T_j) \log_{10} e$, and $j=1,2$. Bayesian decay-parameter estimation in the case of a double-slope decay yields two straight lines corresponding to the two decay slopes, which, in general, will not cross at a point coincident with the turning point of the data (Schroeder curve) and the model decay curve. Rather, the crossing point $P'(x_0, y_0)$ given by
\begin{equation}
x_0 = (a_2 - a_1)/(b_1 - b_2), \quad y_0 = (a_2 b_1 - a_1 b_2)/(b_1 - b_2)
\end{equation}
will generally be lower in level. The turning point $P_t(x_t, y_t)$ is defined to be a point on the decay model curve, to which the crossing point $(P')$ has the minimum distance
\begin{equation}
\sqrt{(x_t - x_0)^2 + (y_t - y_0)^2} \rightarrow \text{min}.
\end{equation}
Two decay times or decay ratio $(T_2/T_1)$, along with the level difference $(\Delta L, \text{in decibel})$ are relevant decay parameters, sufficient for quantifying double-slope characteristics of sound-energy decays. In addition, the estimated coordinate of the turning point $(x_t, y_t)$, particularly, the time instant associated with $x_t$, will approximately show the turning from the first decay process specified by its decay time $T_1$ to the second one.

Successful application of Bayesian analysis to the decomposition of the validated Schroeder decay model [Eq.
Table II. Decay parameters of the Bayesian decay analysis from both measured and diffusion-equation simulated results in the primary room with the receiver at (−2.7, 3, 3) m. Standard deviations Std1 and Std2 associated with decay times $T_1$ and $T_2$ are obtained from Bayesian uncertainty estimations (Ref. 19).

<table>
<thead>
<tr>
<th>Band (kHz)</th>
<th>Data</th>
<th>$T_1$ (s)</th>
<th>Std1</th>
<th>$T_2$ (s)</th>
<th>Std2</th>
<th>Level difference $\Delta L$ (dB)</th>
<th>Turning point (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>Measured</td>
<td>0.32</td>
<td>3.65 $\times 10^{-3}$</td>
<td>0.93</td>
<td>2.17 $\times 10^{-2}$</td>
<td>5.31</td>
<td>86.4</td>
</tr>
<tr>
<td></td>
<td>Simulated</td>
<td>0.32</td>
<td>2.5 $\times 10^{-4}$</td>
<td>0.93</td>
<td>3.31 $\times 10^{-3}$</td>
<td>5.36</td>
<td>50.0</td>
</tr>
<tr>
<td>2.0</td>
<td>Measured</td>
<td>0.31</td>
<td>1.79 $\times 10^{-3}$</td>
<td>0.81</td>
<td>2.83 $\times 10^{-2}$</td>
<td>5.22</td>
<td>86.4</td>
</tr>
<tr>
<td></td>
<td>Simulated</td>
<td>0.32</td>
<td>2.4 $\times 10^{-4}$</td>
<td>0.86</td>
<td>3.55 $\times 10^{-3}$</td>
<td>5.39</td>
<td>60.0</td>
</tr>
<tr>
<td>4.0</td>
<td>Measured</td>
<td>0.29</td>
<td>2.13 $\times 10^{-3}$</td>
<td>0.69</td>
<td>3.87 $\times 10^{-2}$</td>
<td>6.31</td>
<td>91.2</td>
</tr>
<tr>
<td></td>
<td>Simulated</td>
<td>0.30</td>
<td>2.4 $\times 10^{-4}$</td>
<td>0.78</td>
<td>4.54 $\times 10^{-3}$</td>
<td>6.29</td>
<td>50.0</td>
</tr>
<tr>
<td>1.5–4.0 (broad-band)</td>
<td>Measured</td>
<td>0.30</td>
<td>1.27 $\times 10^{-3}$</td>
<td>0.81</td>
<td>2.31 $\times 10^{-2}$</td>
<td>5.64</td>
<td>85.4</td>
</tr>
<tr>
<td></td>
<td>Simulated</td>
<td>0.31</td>
<td>2.4 $\times 10^{-4}$</td>
<td>0.86</td>
<td>2.75 $\times 10^{-3}$</td>
<td>5.75</td>
<td>65.0</td>
</tr>
</tbody>
</table>

The time location of turning points suffers from weaker agreement with experimental values consistently larger than those predicted by the diffusion-equation model. In addition to extra noise and the bumpy nature of measured results as potential sources of errors, the deviations of the measured values from the diffusion-equation modeled ones should be interpreted with caution. While the diffusion-equation model is based on the assumption of sufficient mixing of sound particles, such an assumption is not valid theoretically for a time interval within one mean free time, as pointed out by Morse and Feshbach, and most recently by Valeau et al. In contrast, the acoustic measurements of the room impulse responses contain full information including the direct sound, discrete early reflections, and reverberation tails. Sound-energy propagation within the discrete early-reflection portion cannot be correctly predicted by the diffusion-equation model. In considering the consistently larger turning point times evaluated from experimental results, the differences (especially in 1.5–4 kHz), on an order of two mean free times (2 $\times$ 10.5 ms), may suggest that the diffusion-equation model may be considered valid for the sound-energy prediction after at least two mean free times. The more consistent results from broad-band evaluations of SPL distributions, decay times, and level differences also suggest that the diffusion-equation model can be used in coupled-volume systems for high-frequency, broad-band prediction.

### IV. SOUND-ENERGY FLOWS

According to Fick’s law, the gradient of the sound-energy density $w(r,t)$ at position $r$ and time $t$ in the room under investigation causes the sound-energy flow vector $\mathbf{J}$

$$\mathbf{J} = -D \nabla w(r,t),$$

with $D$ the diffusion coefficient as given in Eq. (1). This section discusses the energy flow in the coupled spaces in terms of diffusion-equation modeling. Emphasis is given to the direction of the energy flow and level of the energy flow as it decays.

#### A. Energy flow directions

This section first discusses the energy flow directions, describing three cases. The natural reverberation times in the primary room are chosen to be smaller, the same as, and larger than the one in the secondary room, respectively (see Table III). The natural reverberation times are reverberation times in either of two rooms when standing alone as a single-room. Absorption coefficients in case 1 are from the mea-
measurement at 2 kHz (see Table I). Absorption coefficients are adjusted for other cases so that the secondary room could be less or equally reverberant than the primary room. The room geometry remains the same as in the previous paragraphs.

A time-dependent solution of the diffusion equation [Eqs. (1), (2), and (3c)] using the source term in Eq. (4) at each observation point in the rooms initially leads to an energy-density impulse response; the energy flow is calculated according to Eq. (13) for each time step (5 ms), termed impulse-response derived energy flow in the following. Figure 9 illustrates the impulse-response derived energy flow directions for several representative time steps. The first time instant is 20 ms, chosen to be on order of two “mean free paths” in the rooms to ensure the validity of the diffusion equation.11,28 The first column of Fig. 9 shows the case where the natural RT in the primary room is smaller, and energy feedback is found around \( t=100 \) ms by tracing the energy flow directions. The energy feedback implies that the sound-energy flows from the secondary room back to the primary room. The feedback energy dominates the decay process in the primary room but with a slower decay rate after \( t=100 \) ms. A double-sloped energy decay is, therefore, expected. For the other two cases, no energy flow has yet been found, indicating that the energy feedback depends on the overall decay rates in both rooms. Most recently Jing and Xiang29 visualized the energy flow directions in form of two- and three-dimensional animations for the three different cases.

The steady-state sound-energy flow-direction changes are obtained by applying Eq. (13) to the steady-state derived energy (density) decay. Figure 10 illustrates the steady-state derived energy flow directions; only the first case \((T'_2 < T''_2)\) is shown since the energy feedback is present in this case. As opposed to the impulse-response derived sound-energy flow, the energy feedback occurs at different times along the steady-state sound-energy flow decay curve.

### B. Energy flow decays

This section studies the energy flow decays. To generate the steady-state derived sound-energy flow decay, an assignment of a switch-off signal to the source term by

\[
q(r, t_s, t) = E_0 \eta(t),
\]

with

\[
\eta(t) = \begin{cases} 1, & t \leq 0 \\ 0, & t > 0 \end{cases},
\]

will yield the steady-state derived energy decay function in solving the diffusion equation [Eqs. (1), (2), and (3c)]. Physically, the sound source is turned on for a long-enough period of time and is then switched off at a time point referred to as 0 ms, the solution of which is called the switch-off energy flow decay. In the numerical implementation, it requires a time-dependent solution already before \( t=0 \) in order to ensure the system arrives at the steady-state. Generally at least twice the computational load is needed in comparison with the time-dependent solution of the energy room impulse response, from which the so-called impulse-response derived sound-energy flow decay is derived. The energy flow level is defined29 as

\[
J_L(r, t) = 10 \log_{10} \left( \frac{\partial w(r, t)}{\partial x} \right)^2 + \left( \frac{\partial w(r, t)}{\partial y} \right)^2 + \left( \frac{\partial w(r, t)}{\partial z} \right)^2 \right)^{1/2}
\]

The diffusion coefficient \( D \) in Eq. (13) is not considered since only the relative amplitude in each room is of major concern.

The energy flow decay shows a flipping-over characteristic in a certain area around the aperture; i.e., when the energy flow amplitude decays to a certain level, it ascends slightly and decays again in a different decay rate. The energy flow decay curve features two points worth mentioning: A “dip” is followed by a “peak.” Figure 11 illustrates a typical energy flow curve obtained at \((-2, 2, 3) \) m and another typical curve without the dip at \((-2, 4, 3) \) m. The first receiver is close to the aperture opening while the second one is not. In this example, the absorption coefficients at 2 kHz are used to assign the boundary conditions for the modified diffusion-equation model.

The dip can be related to the turning point (see Sec. III D). After applying the Bayesian analysis5,26 to the steady-state derived sound-energy decay function, two resulting decay slopes are used to estimate the turning point [see Eq. (12)]. Figure 11(b) illustrates the Bayesian model curve, slope-decomposition, and the turning point. Comparing the dip in Fig. 11(a) with that in Fig. 11(b), the turning point along the time-axis is close to the dip on the energy flow decay curve. The turning point is at \( t=61 \) ms while the dip is at \( t=59 \) ms. So far tested within the scope of the current work, the time occurrence of the turning point on the energy decay curve and that of the dip in the energy flow seem approximately correlated, which has also been found in other receiver locations as long as the feedback energy passes these locations, and dominate the primary energy decay. Table IV lists their specific time instants. In Bayesian analysis, the decay function can be decomposed into two exponential decays terms. The turning point then represents the intersection of two straight decay lines. Both the dip and the turning point indicate the time when the second energy decay starts to dominate the first energy decay. According to Table IV, the dip occurs around the turning point, sometimes slightly before/after the turning point. This is expected to be a result of uncertainties intrinsically residing in Bayesian analysis. The uncertainties quantified by \( \text{Std}_{1,2} \) of decay times \( T_{1,2} \) also imply estimation uncertainties of the turning point.

Figure 12 illustrates the steady-state derived sound-energy flow decay in comparison with the impulse-response
derived sound-energy flow decay at receiver position \((-2, 2, 3)\) m. To obtain the impulse-response derived sound-energy flow, an impulse source signal is used instead. The dips appear at different times, indicating the difference between impulse-response derived decays and steady-state derived decays, and the necessity of using Schroeder-integration for

FIG. 9. (Color online) Two-dimensional mapping of impulse-response derived sound-energy flow vectors [Eq. (13)] for six different snapshots on \(x-y\) plane at \(z=3\) m in the coupled-rooms; the dimension is given in Fig. 1(b). Three different acoustics conditions are characterized by the natural reverberation times \((T^*_1, T^*_2)\) in two rooms. (a) \(T^*_1 < T^*_2\), (b) \(T^*_1 = T^*_2\), and (c) \(T^*_1 > T^*_2\).
the sound-energy decay analysis. From both impulse-
response and steady-state derived sound-energy flow decays,
it is found that the time when the energy flow reverses its
direction is correlated with the time when the dip appears on
the energy flow decay curve. For instance, in the impulse-
response situation, the energy flow at receiver position
$(-2, 2, 3)$ m reverses between $t = 75$ ms and $t = 100$ ms (closer to
$t = 100$ ms) as shown in Fig. 9(a), while the dip appears at 95
ms.

Finally, the physical meaning of the dip in the energy
flow decay curve is explained here. The sound energy ini-
tially flows from the primary room to the secondary room
since the sound-energy density in the primary room is stron-
ger. If the sound energy decays faster in the primary room
than in the secondary room, the change in sign of the energy
gradient across the aperture will, at some future point, indi-
cate flow back to the primary room. This phenomenon mani-
fests itself in magnitude plots such as in the energy flow
decay curve, as a dip [see Fig. 11(a)].

The reversal of energy flow direction is due to the en-
ergy feedback. When feedback dominates the primary energy
decay, flow directions reverse. Physically, the direction changes continuously. Thus, the energy flow magnitude de-
creases gradually until sound-energy flow from the second-
ary room dominates; it reverses direction (a dip appears) and
increases beyond the dip. Eventually, the energy flow reaches
a local peak value [see Fig. 11(a)]; beyond the local peak, the
energy flow decays further. This flow-direction reversal and
the dip in the energy flow decay process cannot exist in cases
in which the primary room’s natural reverberation time is
longer than that of the secondary room. 29

This section has discussed the sound-energy flows deter-
mined using the gradient of the time-dependent sound-
energy density when solving the diffusion equation. Upon
assignment of the source term using Eq. (4), the solution of
the diffusion equation delivers an energy (room) impulse re-
sponse, whose gradient, apart from a constant with a minus
sign, is termed the impulse-response derived sound-energy

FIG. 10. (Color online) Two-dimensional mapping of steady-state derived sound-energy flow vectors [Eq. (13)] for six different snapshots on $x$-$y$ plane at $z = 3$ m in the coupled-rooms; the dimension is given in Fig. 1(b). Overall acoustic condition is characterized by the natural reverberation times $(T_1', T_2')$ in two rooms: $T_1' < T_2'$.  

FIG. 11. (Color online) The normalized energy flow decays and the turning point along the normalized energy decay. (a) Energy flow decay at $(-2, 2, 3)$ m with dip and peak and energy flow decay at $(-2, 4, 3)$ m without dip and peak. (b) Bayesian decay decomposition as well as the turning point of the energy decay curve obtained at $(-2, 2, 3)$ m. The Schroeder decay curve is predicted by the diffusion model while the Bayesian model results from the Bayesian decay-parameter estimation.
flow. Steady-state derived sound-energy decay functions can also be derived upon assignment of the source term using a switch-off function; its gradient is termed, in this section, steady-state derived sound-energy flow. The energy flow-direction reversal is expressed as a dip in the energy flow decay function. The steady-state derived sound-energy flow decays need to be used to correlate the dip with the “turning point” in the Schroeder decay functions, which are the steady-state derived sound-energy decay functions.

V. CONCLUDING REMARKS

This paper discusses previously undiscovered, appealing characteristics predicted by a diffusion-equation model to two coupled spaces. Experimental results are first compared with the simulation results in terms of the SPL distribution along a line parallel and close to the aperture in both the primary room and the secondary room. Experimental results from the sound-energy decay analysis are also compared with the simulation results. Both SPL distributions and energy decay analysis show more consistent results, particularly in some high-frequency bands (1 and 2 kHz) and broadband (1.5–4 kHz) data. Less agreeable results are found when comparing the time instants of turning points between the experimental and modeled results. The consistency of such differences hints at an intrinsic feature of the diffusion-equation model that suggests exclusion by the model of both the direct sound and discrete early reflections. While previous work proves the validity of the diffusion-equation model by showing the SPL distribution along lines across the aperture, this work contributes additional evidence supporting the application of the diffusion-equation model, particularly in high-frequency, broad-band modeling. This work also reveals differences between scale-model measurements and predictions from the diffusion equation when evaluating absolute time occurrence of the turning from the first decay process to the second one.

The time-dependent sound-energy flow is also studied, including the direction changes and the amplitude decay. The diffusion equation is inherently suitable to generate the energy flows that may be derived from the gradient of an efficiently obtained sound-energy density distribution for the room under study. The energy flow-direction changes show the distinct energy feedback when the natural reverberation time in the primary room is smaller than that in the secondary room. The energy flow-direction reversal is expressed as a dip on the energy flow decay curve, which correlates with the turning point on the double-sloped sound-energy decay extracted from the Bayesian analysis. In addition, both impulse response and steady-state, switch-off response are investigated for the time-dependent energy flow, revealing the intrinsic difference between these two responses.

All results discussed in this paper have been restricted to the examples given up to now. Particularly the scale-model experiments within the scope of this study are limited up to 4 kHz in full-scale. A generalization must be carefully examined in future investigations. Future research is expected in following directions: (1) studying the location dependence of double-sloped energy decays with respect to sound source-receiver arrangement relative to the coupling aperture; (2) studying the aperture effect on the energy decay in coupled spaces by systematically changing the aperture size, shape, and location; (3) experimental comparison and verification in real-sized spaces, real concert halls with coupled volumes; (4) refining the valid frequency range/limit for the diffusion-equation model; (5) investigation into the absolute time occurrences of turning points in energy decay in the diffusion-equation model; and (6) experimental verification of energy flows, which may pose experimental challenges.

ACKNOWLEDGMENTS

The authors are grateful to Professor William Siegmann, Professor Joel Plawsky, Dr. Jason Summers, Dr. Vincent Vaileau, Dr. Christopher Jaffe, and Mr. Tomislav Jasa for their stimulating discussions. The authors would like to thank Zu-hre Su and Rolando de la Cruz for their effort in collecting experimental data on the 1:8 scale-models. The authors are also grateful to Associate Editor Professor Lily Wang and the anonymous reviewers whose constructive comments greatly helped in improving clarity from the early version of the manuscript.

TABLE IV. Relationship between the turning point of the sound-energy decay and the dip location along the time-axis of the sound-energy flow decay.

<table>
<thead>
<tr>
<th>Receiver position (m)</th>
<th>Turning point (ms)</th>
<th>Dip (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>−2, 3, 3</td>
<td>63</td>
<td>61</td>
</tr>
<tr>
<td>−1, 3, 3</td>
<td>56</td>
<td>61</td>
</tr>
<tr>
<td>−1, 2.5, 3</td>
<td>56</td>
<td>59</td>
</tr>
<tr>
<td>−2, 2.5, 3</td>
<td>58</td>
<td>58</td>
</tr>
</tbody>
</table>

FIG. 12. Normalized steady-state derived energy flow decay and normalized impulse-response energy flow decay at the same receiver position (−2, 2, 3) m. The steady-state (so-called switch-off) energy flow decay is calculated by assigning the source to be a switch-off function while the impulse-response energy flow decay is calculated by assigning the source an impulsive function.