Abstract: Until now, coprime sensor arrays have used two sparsely spaced subarrays to emulate the performance of a single uniform array with many more sensors (generally on the order of the product of each subarrays’ number of sensors). This allows for similar results with fewer sensors, or the observation of higher frequencies (above the Nyquist limit) with a similar number of sensors. The theory rests on the cross-referencing (using directional filter banks) or cancellation (using product processing) of the $M$ grating lobes in one subarray’s beampattern and $N$ grating lobes in the other, where $M$ and $N$ are coprime integers.

Sets of coprime integers can consist of more than two integers, however, and introducing another coprime factor theoretically multiplies observable frequency (or further decreases the number of array elements needed for the same frequency). Any amount, $n$, of coprime integers and corresponding subarrays may be used. In this work, “$n$-tuple coprime sensor array” theory is expounded and implemented. Experimentally measured beampattern results of a triple coprime sensor array (with three subarrays) are shown, using an extension of the authors’ previously established product processing. Results also confirm that the usable range of an $n$-tuple coprime array extends below its design frequency.

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1. Introduction

Increasingly microphone arrays with a variety of geometries are used for a variety of tasks including traffic noise analysis, sound field reconstruction, and, of course, sound source localization and separation. Such widespread application of microphone arrays fuels a continued, growing interest in sparse and nonuniform array geometries, which hold the promise of offering similar performance using fewer elements (microphones). One such geometry for linear microphone arrays is the so-called coprime sensor array (CSA), as introduced by Vaidyanathan and Pal. CSAs are composed of two uniform linear subarrays which are each sparsely spaced by two separate coprime multiples of the spatial Nyquist limit. This provides $MN$ angular bands using only $M+N$ directional filters (where $M$ and $N$ are the aforementioned coprime factors).

Since the initial introduction of coprime sensing, there have been many efforts to investigate theoretical performance of coprime sensor arrays and corresponding algorithms under various conditions. This includes the authors’ investigations into expanding the coprime theory to include alternate operating frequencies. The results showed that coprime arrays provide useful results at any frequency lower than the design frequency implied by the length and two coprime factors of the standard CSA.

The purpose of this letter is to expand coprime sensing theory to include multiple subarrays (more than two) and to demonstrate the design and processing algorithm using a linear microphone array. Whereas previously a CSA with two subarrays of $M$ and $N$ respective sensors replicates the beam of a uniform linear array (ULA) of $MN$ sensors, a “triple coprime array” with three subarrays of $M$, $N$, and $O$ respective sensors replicates a ULA with $MNO$ sensors. A “quadruple coprime array” would introduce yet another factor. This trend continues for increasing number of subarrays without limit; accordingly, this paper introduces a general term for such extended CSAs (with any number, $n$, subarrays), “$n$-tuple coprime sensor arrays.”

2. Coprime array theory

$n$-tuple coprime arrays are composed of $n$ uniform linear subarrays. One uniform linear array (ULA) with $M$ sensors has a far-field beampattern described by
\[ H(\mathbf{z}(\theta, \lambda)) = \sum_{m=0}^{M-1} h(m) e^{-jam}, \]  

where \( z = (2d/\lambda) \pi \sin \theta \), \( m \) is the individual sensor index, \( \lambda \) is the wavelength, \( j = \sqrt{-1} \), \( h(m) \) is the magnitude or weighting of the sensor \( m \), and \( d \) is the inter-element separation, namely, \( d = LM / N \), where \( L \) is the length. The range of \( \theta \) is restricted to \(-\pi/2 \leq \theta \leq \pi/2\) due to front-back ambiguities in the linear array. Thus the range of \( z \) is \(-\pi \leq z \leq \pi\), implying an lower-limit wavelength of \( 2d \leq \lambda \), or upper-limit frequency of \( f_{ULA} = cM / 2L \), \( \) \( \) \( \) \( \) \( \)

where \( c \) is the wave speed.

Operating such a ULA at higher frequencies introduces aliasing in the form of grating lobes—exact copies of the original beampattern, indistinguishable from one another. At an integer, \( N \), times the limit frequency, a sound source observed by the ULA shows up \( N \) places in the array response (it is “undersampled by a factor of \( N \)”). This aliased beampattern is described by

\[ H(z) = \sum_{m=0}^{M-1} h(m) z^{-Nm}, \]  

where the substitution \( z = \exp(j\pi) \) has been made. If another ULA with different spacing observes the same sound source at the same frequency, but with different grating lobe positions, the “true” direction of the sound source can be determined based on which lobes are shared between the two “subarrays.” To ensure that the grating lobes of each subarray do not coincide in more than one direction, they are aliased by factors which are coprime (sometimes also referred to as mutually prime).

Coprime numbers are integers that do not share any common divisors other than 1. Put another way, coprime numbers do not share any prime factors with one another. The original coprime theory as well as the authors’ experimental acoustic validation of the coprime theory uses two such subarrays, relying only on coprime pairs. However, sets of pairwise coprime numbers can consist of more than just two integers. In fact, the largest set of pairwise coprime numbers would be the same as the set of prime numbers, which is infinite.

The following section develops coprime array theory for three or more subarrays in support of their advantages over standard double coprime arrays with two subarrays.

3. \( n \)-tuple coprime sensing

An \( n \)-tuple coprime sensor array is composed of \( n \) uniform linear subarrays, each spatially undersampled by pairwise coprime factors. For example, a triple coprime array would rely on the set of 3 integers \((M, N, O)\), where \( M, N, \) and \( O \) are pairwise coprime. These integers can be directly used as the “undersampling factors” [analogous to \( N \) in Eq. (3)], or as the number of elements in each subarray, demonstrated in Fig. 1. This work uses the latter convention and the undersampling factor for each subarray is determined by the product of the remaining coprime integers, resulting in the following subarray beampatterns:

\[ H_M(z) = \sum_{m=0}^{M-1} h_M(m) z^{-NMOM}, \]

\[ H_N(z) = \sum_{n=0}^{N-1} h_N(n) z^{-MOOM}, \]  

\[ H_O(z) = \sum_{o=0}^{O-1} h_O(o) z^{-MOM}. \]  

Put another way, the subarray inter-element spacings are \( d_M = NO / 2, \) \( d_N = MO / 2, \) \( d_O = MN / 2 \), which makes the triple coprime design frequency for this array

\[ f_c = cMNO / 2L, \]  

where \( L \) is the shared length of all subarrays. This is a factor of \( O \) higher than the highest frequency observed by the dual coprime array from Sec. 2.
This is easily generalizable to \( n \) subarrays corresponding to the \( n \)-tuple of pairwise coprime integers, \( (M_i)_{i=1}^n \). The beampattern for subarray \( i \) is

\[
H_{M_i}(z) = \sum_{m_i=0}^{M_i-1} h_{M_i}(m_i) z^{-Q_i m_i},
\]  

where \( Q_i \) is the product of each other coprime factor,

\[
Q_i = \prod_{k=1, k \neq i}^n M_k,
\]

and the design frequency for an \( n \)-tuple coprime array with equal-length subarrays is

\[
f_c = \frac{c}{2L} \prod_{i=1}^n M_i.
\]

For the overall \( n \)-tuple coprime sensor array beampattern (irrespective of phase), the absolute value of the product of the subarrays is taken,

\[
\Gamma_{nCSA} = \prod_{i=1}^n |H_{M_i}|,
\]

again, processing in the time domain obviates taking a complex conjugate. With a uniform shading \( (h_{M_i}(m_i) = 1) \), the result can be simplified as a phasor factor (whose absolute value is 1) times a product of sinusoids (for the broadside-facing, fixed beam),

\[
\Gamma_{nCSA}(n) = \frac{\sin \left( \frac{\pi}{2} \prod_{i=1}^n M_i \right)}{\prod_{i=1}^n \sin \left( \frac{\pi}{2} \prod_{k=1, k \neq i}^n M_k \right)}.
\]

4. Experimental methods

This work includes experimental validation of the aforementioned theory. Two \( n \)-tuple coprime arrays, one triple and one quadruple, are constructed and tested in a simulated free-field acoustic environment. The triple coprime array has subarrays of two, three, and five elements, arranged according to Fig. 1, each with 1 m length, providing a design frequency of \( f_c = 5145 \) kHz per Eq. (8). The quadruple coprime array has subarrays of two, three, five, and seven elements, with a length of 1.5 m. The longer aperture was used for the quadruple array in an effort to reduce the design frequency from what would have been 36 kHz down to 24 kHz. Quarter-inch-diameter electret microphones are used in the milled aluminum chassis and medium density fiberboard (MDF) for the triple and quadruple arrays, respectively. Ribbon cables connect the microphones to a 16-channel amplifier with National Instruments data acquisition board. The microphones are individually calibrated with a Larson Davis CAL200 acoustic calibrator.
In order to achieve the necessary source-receiver separation for a far field assumption (impinging waves are approximately planar) and to sufficiently delay room reflections, the experiment was conducted in a large basketball court (36 m by 20 m with a 8 m tall ceiling). The array under test is placed on the floor 9 m from the back wall and the loudspeaker is placed on the floor 18 m away (9 m from the opposite wall). Starting with an end-fire orientation and subsequently turning in 1° increments, 181 impulse response measurements are taken with a logarithmic sine sweep signal and three averages according to Ref. 15.

The resulting impulse response data is truncated with a windowing function prior to room reflections and reverberation to simulate a free-field environment. By processing the overall array output and calculating the root-mean-square value of its truncated 16-channel impulse response at each measurement angle, the conventional fixed beampattern of the array is experimentally validated with 1° resolution (shown in Fig. 2).

The 8 channels of the triple or 14 channels of the quadruple array are processed via the following method. The channels are separated into their respective subarrays. Conventional beamforming is used within each subarray, meaning only the sum of each element is considered (with no delays) and the (181) individual measurements at each angle are used to show the directional response of the array within $-90^\circ \leq \theta \leq 90^\circ$. At this point there is a signal from each subarray at every angle of interest. The subarray signals are then multiplied by one another to get the overall coprime array signal at each angle. To construct the array's directional response, the root-mean-square of each signal is taken, resulting in a positive number for each angle which can be plotted directly in a polar plot (shown in Fig. 2). To show the beampattern at a single frequency, such as the design frequency given by Eq. (8), the impulse responses are convolved with a sine wave prior to processing.

5. Results

After turning the array in 1° increments across the entire $-90^\circ \leq \theta \leq 90^\circ$ range, the microphone signals within each subarray are simply summed, yielding the conventional fixed beampattern in observance of a single sound source. This $l \times 3$ matrix (where $l$ is the length in samples of the windowed free-field impulse response measurement) is convolved with a sine wave at the array’s triple coprime frequency, $f_c = 5.145 \text{ kHz}$. Figure 2(c) shows the experimentally measured, aliased beampatterns of the five-element subarray (long-dashed green), the two-element subarray (short-dashed red), and the three-element subarray (dotted blue). These compare nicely with Fig. 2(a), the computer-simulated subarray beampatterns. Since the triple coprime design frequency is higher than any given pair of subarrays’ dual coprime design frequency, there are multiple angles at which two subarrays’ grating lobes coincide. However, there is only one point

Fig. 2. (Color online) Comparison between predicted and experimentally measured beampatterns of a coprime microphone array with coprime numbers 2, 3, and 5. Experimental results for the 1 m triple coprime array use conventional fixed beamforming (broadside-steered beam) and 181 measurement points ($-90^\circ \leq \theta \leq 90^\circ$) at design frequency of 5145 Hz. All beampatterns are individually normalized for clarity. (a) Predicted beampattern of the three subarrays. (b) Predicted beampattern of the triple coprime microphone array. (c) Experimentally measured beampatterns of the three subarrays. (d) Experimentally measured beampattern of the eight-microphone triple coprime array.
where all three subarray beampatterns overlap completely—the main look angle (0° in this case, using broadside conventional fixed beamforming). These overlapping subarray lobes produce the main beam of the experimentally measured triple coprime array beampattern, shown in Fig. 2(d). This corresponds well to the simulated triple coprime array beampattern for coprime numbers 2, 3, and 5 at its design frequency, shown in Fig. 2(b). Note that the subarrays have been normalized to more easily see them and compare the positions of their grating lobes; their unnormalized amplitudes scale by their number of elements as-expected due to the straightforward addition of microphone signals within each subarray.

6. Discussion

Given eight sensors, the triple coprime sensor array is able to ameliorate aliasing at higher frequencies than a standard double coprime array or uniform linear array of the same length. A quadruple or quintuple (etc.) coprime array would extend that limit even further. Alternatively, at a given upper limit frequency, fewer sensors are required by an \( n \)-tuple coprime array with increased number, \( n \), of subarrays. For example, a 210-element, 3 m uniform linear array with design frequency of 12 kHz could be replaced by a conventional double coprime array with 28 elements, a triple coprime array with 16 elements, or a quadruple coprime array with 14 elements.

It bears repeating that the coprime design frequency for a double coprime array does not represent the only frequency observable by the array. Indeed this holds for the \( n \)-tuple coprime arrays tested in simulations as well as for the triple coprime array tested experimentally. Figure 3 shows the normalized simulated and experimental beampatterns of the subarrays and overall coprime array at a frequency other than the coprime design frequency determined by Eq. (5) [or Eq. (8) for any \( n \)]. This result at \( f = f_c/2 = 2573 \) Hz represents just one example from the range of frequencies (0 ≤ \( f \) ≤ \( f_c \)) that can be observed with the \( n \)-tuple coprime array.

The array can also be steered using straightforward delay-and-sum beamforming in the subarrays prior to their multiplication. In practice, the \( z \) terms in Eqs. (3) and (4) represent time delays in delay-and-sum beamforming, which depend on the angle from broadside of the impinging plane wave. Rather than infer time delay by multiplying by a complex factor in each snapshot to get the array response, one can use the signal value at the corresponding time delay. Since these signals are real and their values at each delay are real, this straightforward delay-and-sum ULA technique in the subarrays obviates taking the complex conjugate of a subarray prior to all of the subarrays’ multiplication.

In their original coprime array work, Vaidyanathan and Pal propose constructing an \( MN \)-band filter bank by combining an \( M \)-band DFT filter bank and an \( N \)-band DFT filter bank. The idea is that only \( M + N \) total shifts are needed to
produce \(MN\) distinct overlaps (there are \(MN\) combinations of the \(M\) and \(N\) shifts). These \(MN\) bands can then be mapped back to the corresponding shifts in each subarray using either a lookup table or the Chinese remainder theorem (CRT). Similarly, in the case of \(n\)-tuple coprime arrays with coprime integers \((M_i)_{i=1}^{n}\), the CRT can uniquely map \(\sum_{i=1}^{n} M_i\) shifts to \(\prod_{i=1}^{n} M_i\) bands.

Just like double coprime arrays, \(n\)-tuple coprime arrays can leverage existing direction of arrival methods that are currently used with uniform linear arrays. Once the algorithm is applied to the limited, unaliased portion of each subarray’s angular range, the CRT or related algorithm\(^{16}\) can be used to infer the direction of arrival in the full range of the coprime array.

7. Conclusion

Using the time-domain beamforming and subarray combination method from Ref. 12, it is possible to incorporate more than two subarrays in a coprime array. Each additional subarray introduced increases the observable frequency of the array by a factor corresponding to its coprime undersampling factor. \(n\) such subarrays compose an \(n\)-tuple coprime array with a design frequency that is necessarily higher than the highest frequency observable by a coprime array of only two subarrays with the same length. Alternatively, at a given frequency and sufficiently large aperture, the number of sensor elements may be drastically reduced as detailed in Sec. 6.

Implementation of an eight-channel, 1 m long triple coprime microphone array with subarrays of two, three, and five microphones validates the \(n\)-tuple coprime array theory for \(n=3\) at the design frequency of 5.15 kHz. Just as with conventional double coprime arrays, grating lobe cancellation of the aliased subarrays occurs at all frequencies lower than the \(n\)-tuple coprime design frequency given by Eq. (8).

Future effort should also be made to answer whether the full breadth of recent coprime sensor array work, particularly in beamforming and direction of arrival algorithms, may be adapted and applied to \(n\)-tuple coprime sensor arrays.

References and links