### Fourier Transform

Given a LTI system: \( h(n) \)

Input a sinusoid (exp.) sequence \( x(n) = e^{j\omega n} \) \( -\infty \leq n \leq \infty \)

\[
y(n) = \sum_{k=-\infty}^{+\infty} h(k) e^{j\omega(n-k)}
\]

\[
= e^{j\omega n} \sum_{k=-\infty}^{+\infty} h(k) e^{-j\omega k} = H(\omega) e^{j\omega n}
\]
Fourier Transform

\[ H(\omega) = \sum_{k=-\infty}^{+\infty} h(k) e^{-j\omega k} \]

- Angular frequency

- Fourier transform of \( h(n) \)

\( H(\omega) \) is a continuous function of \( \omega = 2\pi f \)

\( H(f) \) is periodic over \( \omega + 2\pi \)

Given an arbitrary signal (sequence): \( x(n) \)

Fourier transform:

\[ X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n} \]

- Complex, periodic function of frequency

For time-sequences:

\[ \omega = 2\pi f \]

- Angular frequency

Fourier Transform Pair

Fourier transform:

\[ X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n} \]

Inverse Fourier Transform:

\[ x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \]
Given a LTI system by \( h(n) \):
Response \( y(n) \) -- excitation \( x(n) \)

\[
y(n) = \sum_{k=-\infty}^{+\infty} x(k)h(n - k)
\]

via linear convolution

Fourier transform of \( y(n) \)

\[
Y(\omega) = \sum_{n=-\infty}^{+\infty} y(n)e^{-j\omega n}
\]

\[
y(n) = \sum_{k=-\infty}^{+\infty} x(k)h(n - k)
\]

\[
Y(\omega) = \sum_{n=-\infty}^{+\infty} \left( \sum_{k=-\infty}^{+\infty} x(k)h(n - k) \right)e^{-j\omega n}
\]

\[
Y(\omega) = \sum_{n=-\infty}^{+\infty} \left( \sum_{k=-\infty}^{+\infty} h(n - k)e^{-j\omega (n-k)} \right)
\]

\[
y(n) = \sum_{k=-\infty}^{+\infty} x(k)h(n - k)
\]

\[
Y(\omega) = \sum_{n=-\infty}^{+\infty} \left( \sum_{k=-\infty}^{+\infty} h(n - k)e^{-j\omega (n-k)} \right)
\]

\[
n \rightarrow -\infty: \quad m \rightarrow -\infty \quad m = n - k
\]

\[
n \rightarrow +\infty: \quad m \rightarrow +\infty \quad n = m + k
\]
**Linear Time-Invariant System**

\[
Y(\omega) = \sum_{k=-\infty}^{\infty} x(k) \sum_{m=-\infty}^{\infty} h(m)e^{-jn(\omega+k)} = \sum_{k=-\infty}^{\infty} x(k)e^{-jn\omega} \sum_{m=-\infty}^{\infty} h(m)e^{-jn\omega}
\]

\[
Y(\omega) = \hat{X}(\omega) \cdot \hat{H}(\omega)
\]

**Impulse response**

In time domain:

\[
y(n) \quad h(n) \quad x(n)
\]

In freq. domain:

\[
\hat{Y}(\omega) \quad \hat{H}(\omega) \quad \hat{X}(\omega)
\]

Transfer function

In time domain:

\[
y(n) = h(n) * x(n)
\]

In frequency domain:

\[
\hat{Y}(\omega) = \hat{H}(\omega) \cdot \hat{X}(\omega)
\]

**Discrete Fourier Transform**

**Discrete Fourier Transform**

In time domain:

\[
y(n) = \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k}
\]

In freq. domain:

\[
x(n) = \sum_{k=-\infty}^{\infty} y(k)e^{j\omega k}
\]
**Periodic Sequences**

For periodic sequences \( x(n) = x(n - kN) \)

\[ k \text{ : integer; } n = 0, 1, \ldots, N - 1 \quad N \text{ : period length} \]

**Discrete Fourier Transform**

**Fourier Transform:**

\[ X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n} \]

**For periodic sequences**

\[ X(m) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi \frac{mn}{N}} \]

\( m \): time-sequence with \( N \) number of points

\( X(m) \): complex-valued spectrum (sequence) with the same number of points \( N \)
### Discrete Fourier Transform Pair

**Discrete Fourier transform (DFT)**

\[
X(m) = \sum_{n=0}^{N-1} x(n)e^{-\frac{2\pi}{N}mn}, \quad n, m = 0, 1 \cdots N - 1
\]

**Inverse DFT:**

\[
x(n) = \frac{1}{N} \sum_{m=0}^{N-1} X(m)e^{\frac{2\pi}{N}nm}
\]

---

### Fundamental Features

- **In time-domain:** \( n = 0, 1 \cdots N - 1 \)
  - with a time resolution \( \Delta t = 1 / f_s \)
- **In frequency-domain:** \( m = 0, 1 \cdots N - 1 \)
  - with a frequency-resolution \( \Delta f = f_s / N \)
- Over \( N \) points \( X(m) \) contains content over the entire frequency range until \( f_s \)
- **Only a half** of them \( 0 \leq m < N / 2 \) contains sufficient frequency content of interest

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![Diagram of Discrete Fourier Transform Pair](image-url)
**Fundamental Features**

- **Periodic** time-domain signals lead to **discrete** Fourier Spectrum.

- In frequency-domain: **discrete** Fourier spectrum is also a **periodic** spectrum with the same period length.

- **Periodic** Fourier spectrum leads to **discrete** time-domain sequence.

- A single finite-length sequence can be considered as ‘**periodic**’ when using DFT.

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**Fundamental Features**

Given a time-domain sequence (signal) and its **discrete** Fourier Spectrum: \( x(n) \) \( \rightarrow \) \( X(m) \)

Then: \( x(-n) \) \( \rightarrow \) \( X(-m) \)

- If the time-domain sequence is real-valued, then: \( x(-n) \) \( \rightarrow \) \( X^*(m) \)

(see Assignment 7-8 below)

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**Fundamental Features**

- **Impulse response**

  In time domain: \( y(n) = \sum h(n) \cdot x(n) \)

  In freq. domain: \( Y(m) = H(m) \cdot X(m) \)

- **Transfer function**

  In time domain: \( y(n) = h(n) \ast x(n) \)

  In frequency domain: \( Y(m) = H(m) \cdot X(m) \)
### Convolution/correlation

- **Convolution**
  \[ x(n) * y(n) = \sum_{k=0}^{N-1} x(k) y(n-k) \]

- **(Cross-)correlation**
  \[ x(n) * y(-n) = x(n) \otimes y(n) = \sum_{k=0}^{N-1} x(k) y(n+k) \]

- **Cross-correlation**
  \[ g_{xy}(n) = x(n) \otimes y(n) = x(n) * y(-n) \]

- **Auto-correlation**
  \[ g_{xx}(n) = x(n) \otimes x(n) = x(n) * x(-n) \]

### Cross-Spectrum

- **Cross-spectrum** → Fourier Transform of cross-correlation
  \[ g_{xy}(n) \rightarrow G_{XY}(m) \]

- **Cross-correlation of real-valued time sequences**
  \[ g_{xy}(n) = x(n) \otimes y(n) = x(n) * y(-n) \]

- **Cross spectrum**
  \[ G_{XY}(m) = X(m) Y^*(m) \]

- **Auto-spectrum**
  \[ G_{XX}(m) = X(m) X^*(m) = |X(m)|^2 \]

### Summary

- **Fourier transforms**
  Spectrum in frequency domain

- **Input/output relation of a LTI system:**
  - In time domain: linear convolution
  - In frequency domain: multiply the transfer function by the input spectrum

- **Time-domain convolution ↔ Spectral multiplication**

- **Discrete Fourier transform**
Assignment #2

1. A time sequence $d(n)$ is given, determine its spectrum $D(\omega)$ in terms of Fourier Transform, and using that detailed expression to show: $D(\omega) = D(\omega + 2\pi)$

2. The response of a LTI system can be expressed by a linear convolution $y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k)$
   
   Apply Fourier Transform $Y(\omega) = \sum_{n=-\infty}^{\infty} y(n)e^{-j\omega n}$ to $y(n)$
   
   as stated above, to prove:
   $$Y(\omega) = H(\omega) \cdot X(\omega)$$

3. Calculate the Fourier transform of a unit sample sequence $\delta(n)$

4. A time signal $x(n) = -5 \cdot \delta(n)$ is given, calculate its Fourier Transform $X(\omega)$. Sketch and discuss the result of $X(\omega)$ as function of angular frequency.

5. A periodic signal $y(n) = 4.2 \cdot \delta(n-k)$ has its period length $N$ where $k < N$ and $k$ is fixed. Calculate its discrete Fourier transform, and sketch the results of its magnitude and phase as function of frequency.

6. Show (prove)
   $$\delta(n) \ast x(n) = x(n)$$

   Convolution of $\delta(n)$ with any sequence $x(n)$

   results in $x(n)$, the sequence itself (unchanged)!
7. Given discrete Fourier transform $X(m)$ of a periodic time sequence $x(n)$ with a period of $M$, show that the time-reversed sequence $x(-n)$ has a discrete Fourier transform $X(-m)$ namely

$$x(-n) \rightarrow X(-m)$$

8. If the periodic time sequence $x(n)$, is real-valued, show

$$x(-n) \rightarrow X^*(m)$$